Randomness & Complexity in Pure Mathematics


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Abstract. One normally thinks that everything that is true is true for a reason. I've found mathematical truths that are true for no reason at all. These mathematical truths are beyond the power of mathematical reasoning because they are accidental and random.

Using software written in Mathematica that runs on an IBM RS/6000 workstation, I constructed a perverse 200-page algebraic equation with a parameter $N$ and 17,000 unknowns:

$$\text{Left-Hand-Side}(N) = \text{Right-Hand-Side}(N).$$

For each whole-number value of the parameter $N$, ask whether this equation has a finite or an infinite number of whole number solutions. The answers escape the power of mathematical reason because they are completely random and accidental.

This work is an extension of famous results of Gödel and Turing using ideas from a new field called algorithmic information theory.

1 Hilbert on the axiomatic method

Last month I was a speaker at a symposium on reductionism at Cambridge University where Turing did his work. I'd like to repeat the talk I gave there and explain how my work continues and extends Turing's. Two previous speakers had said bad things about David Hilbert. So I started by saying that in spite of what you might have heard in some of the previous lectures, Hilbert was not a twit!

Hilbert's idea is the culmination of two thousand years of mathematical tradition going back to Euclid's axiomatic treatment of geometry, going back to Leibniz's dream of a symbolic logic and Russell and Whitehead's monumental Principia Mathematica. Hilbert's dream was to once and for all clarify the methods of mathematical reasoning. Hilbert wanted to formulate a formal axiomatic
system which would encompass all of mathematics.

\textbf{Formal Axiomatic System}
\[
\rightarrow \text{ consistent} \\
\rightarrow \text{ complete} \\
\rightarrow
\]

Hilbert emphasized a number of key properties that such a formal axiomatic system should have. It's like a computer programming language. It's a precise statement about the methods of reasoning, the postulates and the methods of inference that we accept as mathematicians. Furthermore Hilbert stipulated that the formal axiomatic system encompassing all of mathematics that he wanted to construct should be “consistent” and it should be “complete.”

\textbf{Formal Axiomatic System}
\[
\rightarrow \text{ consistent} \\
\rightarrow \text{ complete} \\
\rightarrow
\]

Consistent means that you shouldn't be able to prove an assertion and the contrary of the assertion.

\textbf{Formal Axiomatic System}
\[
\rightarrow \text{ consistent } A \not\rightarrow \neg A \\
\rightarrow \text{ complete } A \not\rightarrow \neg A \\
\rightarrow
\]

You shouldn't be able to prove \( A \) and not \( A \). That would be very embarrassing.

Complete means that if you make a meaningful assertion you should be able to settle it one way or the other. It means that either \( A \) or not \( A \) should be a theorem, should be provable from the axioms using the rules of inference in the formal axiomatic system.

\textbf{Formal Axiomatic System}
\[
\rightarrow \text{ consistent } A \not\rightarrow \neg A \\
\rightarrow \text{ complete } A \not\rightarrow \neg A \\
\rightarrow
\]

Consider a meaningful assertion \( A \) and its contrary not \( A \). Exactly one of the two should be provable if the formal axiomatic system is consistent and complete.

A formal axiomatic system is like a programming language. There's an alphabet and rules of grammar, in other words, a formal syntax. It's a kind of thing that we are familiar with now. Look back at Russell and Whitehead's three enormous volumes full of symbols and you'll feel you're looking at a large computer program in some incomprehensible programming language.

Now there's a very surprising fact. Consistent and complete means only truth and all the truth. They seem like reasonable requirements. There's a funny consequence, though, having to do with something called the decision problem. In