4. Abstract and concrete theory

The theory of Abstract Julia sets and of the Abstract Mandelbrot set was developed in a self-contained manner. We now illustrate this theory in the complex plane. In particular, we complete verification of the statements given in Chapter 1. The whole discussion follows the motto ‘Look at the abstract models, and ask for what remains true in the complex plane’.

4.1 Quadratic iteration

4.1.1 Julia equivalences and Julia sets

We continue the listing of well-known statements on quadratic polynomials started in Chapter 1.1.3. Much of the following is valid beyond the case of quadratic polynomials, but we do not need the general statements.

V. Classification of Fatou components. For a given $c \in \mathbb{C}$, the complement of the Julia set $J_c$ in $\bar{\mathbb{C}}$ is said to be the Fatou set of $p_c$, and its connectedness components are called the Fatou components. A Fatou component either contains the fixed point $\infty$ or is a subset of the filled-in Julia set. In the latter case we call it bounded.

Each non-repelling periodic point $z \in \mathbb{C}$ of some period $m$ belongs to the closure of a bounded Fatou component. Namely, for $g = p_c^m$ the following holds:

1. If the point $z$ is attractive, then it lies in a bounded Fatou component $K$, which is $g$-invariant. For all points $x \in K$ it holds $\lim_{n \to \infty} g^n(x) = z$. Therefore, $K$ is called the immediate basin of $z$.

2. If $z$ is parabolic, there exist a $q \in \mathbb{N}$ and mutually different bounded Fatou components $K_1, K_2, \ldots, K_q = K_0$ satisfying the following properties: For all $j = 1, 2, \ldots, q$ it holds $g(K_{j-1}) = K_j, z \in \partial K_j \subseteq J_c$ and $\lim_{n \to \infty} g^n(x) = z$ if $x \in K_j$. (Note that all sets $p_c^n(K_0); n = 1, 2, 3, \ldots, mq$ are mutually disjoint.) Each $K_j; j = 1, 2, \ldots, q$ is called an immediate basin of $z$. 
3. If \( z \) is irrationally indifferent, then either \( z \in J_c \) or \( z \in K_c \setminus J_c \). In the first case \( z \) is called a Cremer point. In the second case \( z \) belongs to a bounded Fatou component \( K \) on which \( g \) is conformally conjugate to an irrational rotation with rotation number \( \nu \) (relative to \( 2\pi \)) given by \( (p_c^n)'(z) = e^{2\pi i \nu} \): there exists a conformal bijection \( \phi \) from \( K \) onto the open unit disk satisfying \( g(x) = \phi^{-1}((p_c^n)'(z)\phi(x)) \) for all \( x \in K \). The set \( K \) is called a Siegel disk, and the point \( z \) a Siegel point.

Clearly, in each case \( K \) is periodic, but the fact that the orbit of each bounded Fatou component contains the immediate basin of some attractive or parabolic periodic point or some Siegel disk is highly non-trivial. It follows from Sullivan's famous theorem on non-wandering domains: each Fatou component of a rational map is preperiodic or periodic (see [163]).

VI. Local connectivity of some Julia sets. For simplicity, we call a parameter \( c \in M \) hyperbolic (superattractive, parabolic, irrationally indifferent) of period \( m \) if \( p_c \) has an attractive (superattractive, parabolic, irrationally indifferent) orbit of period \( m \). By the multiplier of such a parameter \( c \) we mean the multiplier of the corresponding periodic orbit. (Recall that there exists at most one non-repelling orbit for some \( p_c \).)

If a parameter \( c \) is hyperbolic or parabolic, or if \( c \) is preperiodic with respect to \( p_c \), then \( J_c \) is locally connected (see [43, 114, 25, 166]). This will be of high importance in the following discussion. Note that parameters \( c \) being preperiodic with respect to \( p_c \) are called Misiurewicz points.

VII. Periodic and preperiodic points depending on the parameter. In Chapter 1.2.2 we discussed the dependence of periodic points on the parameter. Let us generalize the discussion to the preperiodic case. For this, fix some pair \( (c_0, z_0) \), where \( z_0 \) is a preperiodic point of preperiod \( n \) and period \( m \) for some \( p_{c_0}; c_0 \in \mathbb{C} \). \((n = 0 \text{ provides the periodic case.})\]

Clearly, the map \( g \) defined by \( g(c, z) = p_c^{m+n}(z) - p_c^n(z) \) satisfies \( g(c_0, z_0) = 0 \), and we have

\[
\frac{\partial g}{\partial z}(c_0, z_0) = ((p_{c_0}^m)'(p_{c_0}^n(z_0)) - 1)2^n \prod_{i=0}^{n-1} p_{c_0}^i(z_0).
\]

The term \( (p_{c_0}^m)'(p_{c_0}^n(z_0)) \) is no more than the multiplier of the \( n \)-th iterate of \( z_0 \), and so \( \frac{\partial g}{\partial z}(c_0, z_0) \) is equal to 0 iff this iterate is parabolic of multiplier 1 or if 0 lies on the orbit of \( z_0 \). (Later it will turn out that both at the same time is impossible.)

Otherwise, the implicit function theorem defines a locally invertible holomorphic map \( z = z(c) = z_{(c_0, z_0)}(c) \) in some neighborhood \( U \) of \( c_0 \) assigning to \( c \in U \) a preperiodic point of preperiod \( n \) and period \( m \) and satisfying \( z_{(c_0, z_0)}(c_0) = z_0 \).