Part I

Toolbox for oscillatory integrals
2. "Exponential tools" for evaluating oscillatory integrals

This chapter is devoted to the study of problem 1.1.4, i.e. to the computation of upper bounds and equivalents of oscillatory integrals involving solutions of nonlinear analytic differential equations.

2.1 Mono-frequency oscillatory integrals: complexification of time

In this section we define spaces of functions which enable us to compute simultaneously equivalents of the mono-frequency oscillatory integrals

\[ I^{\pm}(f, \varepsilon) = \int_{-\infty}^{+\infty} f(t, \varepsilon)e^{\pm \omega t/\varepsilon} dt. \]

Moreover we explain how to prove that solutions of nonlinear differential equations belong to these spaces.

All these spaces are composed of holomorphic functions defined in domains symmetric with respect to the real axis. If one is only interested in the computation of an equivalent of \( I^+ \) (resp. \( I^- \)), it is sufficient to work with functions defined on a half domain, i.e. holomorphic for \( \Im \xi \geq 0 \) (resp. holomorphic for \( \Im \xi \leq 0 \)). However, in most of the cases, the functions are real on the real axis. Thus, their holomorphic continuation is symmetric with respect to the real axis and hence, automatically defined on domains symmetric with respect to the real axis.

2.1.1 Rough exponential upper bounds

We begin with a very simple lemma which gives exponential upper bounds for mono-frequency oscillatory integrals.

\[ \textbf{Lemma 2.1.1 (First Mono-Frequency Exponential Lemma).} \]

\textit{Let} \( \omega, \ell, \lambda \text{ be three real positive numbers. Let } B_\varepsilon \text{ be the strip in the complex field: } B_\varepsilon = \{ \xi \in \mathbb{C} / |\Im(\xi)| < \ell \}. \]

\textit{Let} \( H_\lambda^\varepsilon \text{ be the set of functions } f \text{ satisfying} \]

\[ \frac{\omega}{(\lambda t)^\varepsilon} e^{-\lambda t} f(t, \varepsilon) \text{ is bounded on } B_\varepsilon. \]

\[ I^{\pm}(f, \varepsilon) \leq C \int_{B_\varepsilon} |f(t, \varepsilon)| \frac{\omega}{(\lambda t)^\varepsilon} e^{-\lambda t} dt. \]