Self-Duality in Nonlinear Electromagnetism

M.K. Gaillard and B. Zumino

Physics Department, University of California, and
Theoretical Physics Group, Lawrence Berkeley National Laboratory

Abstract. We discuss duality invariant interactions between electromagnetic fields and matter. The case of scalar fields is treated in some detail.

1 Duality Rotations in Four Dimensions

The invariance of Maxwell's equations under "duality rotations" has been known for a long time. In relativistic notation these are rotations of the electromagnetic field strength $F_{\mu\nu}$ into its dual, which is defined by

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} F^{\lambda\sigma}, \quad \tilde{F}_{\mu\nu} = -F_{\mu\nu}.$$  (1)

This invariance can be extended to electromagnetic fields in interaction with the gravitational field, which does not transform under duality. It is present in ungauged extended supergravity theories, in which case it generalizes to a nonabelian group (Ferrara et al. (1977), Cremmer, Julia (1979)). In (Gaillard, Zumino (1981), Zumino (1982)) we studied the most general situation in which duality invariance of this type can occur. More recently (Gibbons, Rasheed (1995)) the duality invariance of the Born–Infeld theory, suitably coupled to the dilaton and axion (Gibbons, Rasheed (1996)), has been studied in considerable detail. In the present note we will show that most of the results of (Gibbons, Rasheed (1995), Gibbons, Rasheed (1996)) follow quite easily from our earlier general discussion. We shall also present some new results that were not made explicit in (Gaillard, Zumino (1981), Zumino (1982)), especially some properties of the scalar fields.

We begin by recalling and completing some basic results of our paper (Gaillard, Zumino (1981), Zumino (1982)). Consider a Lagrangian which is a function of $n$ real field strengths $F_{\mu\nu}^a$ and of some other fields $\chi^i$ and their derivatives $\chi_{\mu}^i = \partial_{\mu} \chi^i$:

$$L = L (F^{a}_{\mu\nu}, \chi^i, \chi_{\mu}^i) .$$  (2)

Since

$$F_{\mu\nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a ,$$  (3)

we have the Bianchi identities

$$\partial_\mu \tilde{F}^a_{\mu\nu} = 0 .$$  (4)

On the other hand, if we define
we have the equations of motion
\[ \partial^\mu \tilde{G}^a_{\mu\nu} = 0. \] (6)

We consider an infinitesimal transformation of the form
\[ A \cdot x = x', \] (7)
where \( A, B, C, D \) are real \( n \times n \) constant infinitesimal matrices and \( \xi^i(\chi) \) functions of the fields \( \chi^i \) (but not of their derivatives), and ask under what circumstances the system of the equations of motion (4) and (6), as well as the equation of motion for the fields \( \chi^i \) are invariant. The analysis of (Gaillard, Zumino (1981)) shows that this is true if the matrices satisfy
\[ A^T = -D, \quad B^T = B, \quad C^T = C, \] (9)
which the superscript \( T \) denotes the transposed matrix) and in addition the Lagrangian changes under (7) and (8) as
\[ \delta L = \frac{1}{4} \left( F c \bar{F} + G b \bar{G} \right). \] (10)

The relations (9) show that (7) is an infinitesimal transformation of the real noncompact symplectic group \( \text{Sp}(2n, R) \) which has \( \text{U}(n) \) as maximal compact subgroup. The finite form is
\[ \left( \begin{array}{c} F' \\ G' \end{array} \right) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} F \\ G \end{array} \right), \] (11)
where the \( n \times n \) real submatrices satisfy
\[ c^T a = a^T c, \quad b^T d = d^T b, \quad d^T a - b^T c = 1. \] (12)

Notice that the Lagrangian is not invariant. In (Gaillard, Zumino (1981)) we showed, however, that the derivative of the Lagrangian with respect to an invariant parameter \( is \) invariant. The invariant parameter could be a coupling constant or an external background field, such as the gravitational field, which does not change under duality rotations. It follows that the energy-momentum tensor, which can be obtained as the variational derivative of the Lagrangian with respect to the gravitational field, is invariant under duality rotations. No explicit check of its invariance, as was done in (Gibbons, Rasheed (1995), Gibbons, Rasheed (1996), Born, Infeld (1934), Schrödinger (1935)), is necessary.

The symplectic transformation (11) can be written in a complex basis as