Abstract: As a first approximation, a metal can be modelled as an electron gas. A non-interacting electron gas has a continuous spectrum of electron-hole pair excitations. At each wavevector $Q$ with $|Q|$ less than the maximum Fermi surface spanning vector $(2k_F)$ there is a continuous set of electron-hole pair states, with a maximum energy but no gap (the minimum energy is zero.) Once the Coulomb interaction is taken into account, a new collective mode, the plasmon, is built from the electron-hole pair spectrum. The plasmon captures most of the spectral weight in the scattering cross-section, yet the particle-hole pairs remain practically unchanged, as can be seen from the success of the Landau Fermi-liquid picture. This article explores how even an isolated electron-hole pair in non-interacting approximation is a form of charge density wave excitation, and how the Coulomb interaction totally alters the charge properties, without affecting many other properties of the electron-hole pairs.

1 Introduction

The low-lying excitations of a non-interacting electron gas are simple rearrangements of the occupancy of the single electron plane-wave orbitals. Because real metals, according to Landau theory, have a lot in common with non-interacting quantum electron gases, this subject is well known to all who study solids. If we neglect band structure, the electron orbitals (labeled by quantum numbers $k = (k, \sigma)$) have energy $\epsilon_k = \hbar^2 k^2 / 2m$. The ground state has all orbitals occupied which lie inside the Fermi surface, with wavevectors obeying $|k| < k_F$ and energies obeying $\epsilon_k < \epsilon_F$, where $k_F$ and $\epsilon_F$ are the Fermi wavevector and energy. The simplest excitation, known as an electron-hole (e-h) pair, consists of moving an electron out of a state $k$ below the Fermi surface, putting it into a state $k + Q$ above. This is shown in Fig. (1).

The Coulomb interaction is numerically not small, but Landau theory argues that nevertheless, the consequences of the Coulomb interaction are less drastic than one might suppose. However, one drastic effect certainly happens, namely, out of the e-h pair excitation spectrum, the Coulomb interaction creates a new,
collective excitation, the plasmon. This typically lies at quite a high energy. Fig. (2) shows the spectrum predicted in random phase approximation (RPA) for sodium metal. The plasmon has an energy $\hbar \omega_p(Q)$ which starts at $\approx 6$ eV at $Q = 0$ and disperses upwards in energy. Also shown in this picture is the continuous spectrum of e-h pairs. This spectrum is easily understood from Fig. (1). At any fixed value of $Q$ with $|Q| < 2k_F$, it is possible to find orbitals $\mathbf{k}$ just below the Fermi surface such that the corresponding orbital $\mathbf{k} + Q$ lies just above the Fermi surface. This means that pair excitations exist with arbitrarily small excitation energies for $Q < 2k_F$, whereas for larger $Q$, a gap exists to the lowest single pair excitation. For any $Q$ there is also a maximum energy e-h pair which can be created, namely when the hole state $\mathbf{k}$ lies just below the Fermi surface in the direction of $Q$, and the corresponding electron state then lies as far as possible outside the Fermi sea. This excitation has energy $\hbar^2(k_F + Q)^2/2m - \hbar^2k_F^2/2m$. This formula gives the upper edge of the e-h pair continuum shown in Fig. (2). Surprisingly, in spite of the totally new plasmon found in RPA, nevertheless, the spectrum of e-h pairs is not altered from the free electron value in RPA.

This subject has been understood for more than 30 years. Nevertheless, the standard treatments in solid state and most many-body texts [1] do not discuss certain aspects which I find paradoxical. This article is intended as pedagogical, aiming to state and then to explain these paradoxes as clearly as possible. Of course, there is no actual paradox in the existing theory, which provides successful approximate methods for calculating many properties of metals, but the most popular approaches use language in which these interesting paradoxical issues are never apparent.

The paradoxes are forcefully apparent in two very interesting experimental