Convergence Characteristics of Approximate Factorization Methods

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Abstract. Convergence characteristics and efficiency of several implicit approximate factorization schemes (including standard ADI, a diagonally dominant form of ADI-DDADI and a diagonalized version of DDADI:D3ADI) are examined using stability analysis and numerical convergence studies. We also discuss the use of sub-iterations to eliminate the approximate factorization errors and thereby improve convergence behavior. Numerical convergence studies are utilized to demonstrate the theoretical findings. We conclude that, ADI and D3ADI with sub-iterations perform equally well in terms of efficiency, although D3ADI may be more robust with respect to time step selection.

1 Introduction

Convergence characteristics and efficiency of several implicit approximate factorization schemes are examined using stability analysis and numerical convergence studies. The standard alternating direction implicit (ADI) procedure of Beam and Warming (76) is used as the baseline method. We consider two variants of the ADI method here—(1) the DDADI scheme, or the diagonally-dominant ADI scheme, originally devised by Bardina and Lombard (87) and recently resurrected by MacCormack (97), and (2) the D3ADI scheme, developed by Klopfer, et. al. (98) as an extension of the DDADI scheme to the diagonalized-ADI scheme of Pulliam and Chaussee (81). In addition, the use of sub-iterations (see, for example, Venkateswaran, Buelow and Merkle (97)), to eliminate the approximate factorization errors is examined as a means of improving convergence behavior.

The stability analysis is performed on the two-dimensional linearized Euler equations in full system form. Third-order Roe upwind differencing is used for the right-hand side (RHS) residual evaluation, while generally first-order upwind differencing is used for the implicit operator on the left-hand side (LHS). All terms are evaluated at constant fixed-point values of the flow variables. Fourier stability analysis is performed for the various factorization approximations and the resulting linear eigenvalue problem is solved numerically (see Jespersen and Pulliam (83)). Finally, numerical convergence studies are utilized to demonstrate the theoretical findings.
2 Theoretical Development

The schemes considered here each possess distinct linearization and approximate factorization error terms, which, in turn, determine the stability and convergence behavior of the algorithm. In the interest of brevity, we do not present all the details of the numerical analysis. The major conclusions of the present study may be summarized as follows:

**Direct scheme: UNF.** The "delta form" of the unfactored implicit $\theta$ scheme applied to the linearized Euler equations produces

$$[I + h\theta T_x^n + h\theta T_y^n] \Delta Q^n = -hR^n$$

where $Q$ is the vector of solution variable and $R$ the residual operator formed using the third-order accurate flux split upwind scheme. In Eq. 1, $h = \Delta t$ and $\theta = 1$ for 1st Order Euler implicit. The operators $T$ represent block tri-diagonal operators in each coordinate direction, producing a large sparse multidimensional LHS system which is computationally expensive. As expected, a consistent (3rd/3rd) scheme has an amplification factor $\sigma \to 0$ as CFL $\to \infty$. On the other hand, inconsistent linearization (first order accuracy on the left-hand side and third-order accuracy on the right-hand side) leads to a 'freezing' of the amplification factor spectrum for large time-step sizes even when the implicit operator is solved directly (i.e., no approximate factorization). Consequently, although the scheme is unconditionally stable, there is little convergence benefit to be gained from running at infinite CFL numbers.

**Approximate Factorization: ADI.** Standard approximate factorization (ADI), replaces the LHS of Eq. 1 with the two factor scheme

$$[I + h\theta T_x][I + h\theta T_y] \Delta Q^n = -hR^n$$

where now each factor on the LHS is a one-dimensional block tri-diagonal system which is more efficient to solve. The penalty for this efficiency approximation is the neglected cross term error

$$\mathcal{E}_{ADI} = h^2\theta^2 T_x T_y \Delta Q^n$$

In the case of time accuracy, the error behaves as $O(h^3)$ for small $h$, ($\Delta Q^n$ is $O(h)$ for small $h$) which is a second order error term. In terms of steady-state application, for large $h$, we assume $\Delta Q^n$ is $O(1)$ and the cross term behaves as $O(h^2)$. The standard ADI scheme yields the well-known result that $\sigma \to 1$ as CFL $\to \infty$. This is a consequence of the cross-term factorization errors on the left-hand side. As a result, there is an optimal range of CFL number, for which the ADI scheme produces reasonable convergence efficiency. Typically, the optimal value is between 5 and 10. It also has been shown that the diagonalized-ADI (DADI) scheme of Pulliam and Chaussee (81) possesses the same convergence characteristics as the standard procedure.