Between Sobolev and Poincaré

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Abstract. Let \( a \in [0, 1] \) and \( r \in [1, 2] \) satisfy relation \( r = 2/(2 - a) \). Let \( \mu(dx) = c_r \exp(-(|x_1|^{r} + |x_2|^{r} + \ldots + |x_n|^{r}))dx_1dx_2\ldots dx_n \) be a probability measure on the Euclidean space \((\mathbb{R}^n, \| \cdot \|)\). We prove that there exists a universal constant \( C \) such that for any smooth real function \( f \) on \( \mathbb{R}^n \) and any \( p \in [1, 2) \)

\[
\mathbb{E}_\mu f^2 - (\mathbb{E}_\mu f^p)^{2/p} \leq C(2 - p)^{a-1} \mathbb{E}_\mu \| \nabla f \|^2.
\]

We prove also that if for some probabilistic measure \( \mu \) on \( \mathbb{R}^n \) the above inequality is satisfied for any \( p \in [1, 2) \) and any smooth \( f \) then for any \( h : \mathbb{R}^n \to \mathbb{R} \) such that \( |h(x) - h(y)| \leq \|x - y\| \) there is \( \mathbb{E}_\mu |h| < \infty \) and

\[
\mu(h - \mathbb{E}_\mu h > \sqrt{C} \cdot t) \leq e^{-Kt^{r}}
\]

for \( t > 1 \), where \( K > 0 \) is some universal constant.

Let us begin with a few definitions.

Definition 1. Let \((\Omega, \mu)\) be a probability space and let \( f \) be a measurable, square integrable non-negative function on \( \Omega \). For \( p \in [1, 2) \) we define the \( p \)-variance of \( f \) by

\[
\text{Var}(p)_{\mu}(f) = \int_{\Omega} f(x)^p \mu(dx) - \left( \int_{\Omega} f(x)^p \mu(dx) \right)^{2/p} = \mathbb{E}_\mu f^2 - (\mathbb{E}_\mu f^p)^{2/p}.
\]

Note that \( \text{Var}(1)_{\mu}(f) = D^2_{\mu}(f) = \text{Var}_{\mu}(f) \) coincides with classical notion of variance, while

\[
\lim_{p \to 2^-} \frac{\text{Var}(p)_{\mu}(f)}{2 - p} = \frac{1}{2} \left( \mathbb{E}_\mu f^2 \ln(f^2) - \mathbb{E}_\mu f^2 \cdot \ln(\mathbb{E}_\mu f^2) \right) = \frac{1}{2} \text{Ent}_{\mu}(f^2),
\]

where \( \text{Ent}_{\mu} \) denotes a classical entropy functional (see [L] for a nice introduction to the subject).

Definition 2. Let \( \mathcal{E} \) be a non-negative functional on some class \( C \) of non-negative functions from \( L^2(\Omega, \mu) \). We will say that \( f \in C \) satisfies

- the Poincaré inequality with constant \( C \)

if \( \text{Var}_{\mu}(f) \leq C \cdot \mathcal{E}(f) \),

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the logarithmic Sobolev inequality with constant $C$
if $\text{Ent}_\mu(f^2) \leq C \cdot \mathcal{E}(f)$,
the inequality $I_\mu(a)$ (for $0 \leq a < 1$) with constant $C$
if $\text{Var}(p)_\mu(f) \leq C \cdot (2 - p)^a \cdot \mathcal{E}(f)$ for all $p \in [1, 2)$.

Lemma 1. For a fixed $f \in \mathcal{C}$ and $p \in [1, 2)$ let

$$
\varphi(p) = \frac{\text{Var}(p)_\mu(f)}{1/p - 1/2}.
$$

Then $\varphi$ is a non-decreasing function.

Proof. Hölder’s inequality yields that $\alpha(t) = t \ln(E_\mu f^{1/t})$ is a convex function
for $t \in (1/2, 1]$. Hence also $\beta(t) = e^{2\alpha(t)} = (E_\mu f^{1/t})^{2t}$ is convex and therefore
$\beta(t) - \beta(1/2)$ is non-decreasing on $(1/2, 1]$. Observation that

$$
\varphi(p) = \frac{\beta(1/2) - \beta(1/p)}{1/p - 1/2}
$$

completes the proof.

Corollary 1. For $f \in \mathcal{C}$ the following implications hold true:

- $f$ satisfies the Poincaré inequality with constant $C$
  if and only if $f$ satisfies $I_\mu(0)$ with constant $C$,
- if $f$ satisfies the logarithmic Sobolev inequality with constant $C$
  then $f$ satisfies $I_\mu(1)$ with constant $C$,
- if $f$ satisfies $I_\mu(1)$ with constant $C$
  then $f$ satisfies the logarithmic Sobolev inequality with constant $2C$,
- if $f$ satisfies $I_\mu(a)$ with constant $C$ and $0 \leq a \leq 1$
  then $f$ satisfies $I_\mu(a)$ with constant $C$.

Proof.

- To prove the first part of Corollary 1 it suffices to note that $p \mapsto \text{Var}(p)_\mu(f)$ is a non-increasing function.
- The second part of Corollary 1 follows easily from the fact that

$$
\lim_{p \to 2^-} \frac{\text{Var}(p)_\mu(f)}{2 - p} = \frac{1}{2} \cdot \text{Ent}_\mu(f^2).
$$

- To prove the third part of Corollary 1 use Lemma 1 and note that for $p \in [1, 2)$ we have

$$
\frac{\text{Var}(p)_\mu(f)}{2 - p} = \frac{\varphi(p)}{2p} \leq \lim_{p \to 2^-} \frac{\varphi(p)}{2} = \text{Ent}_\mu(f^2).
$$

- The last part of statement is trivial.