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Weakly Positive Continuous-Time
Linear Systems

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1 Introduction

The analysis of singular (descriptor) discrete-time and continuous-time system has been considered in many papers and books [1,2,5,10-14]. Properties of the fundamental matrix of singular discrete-time linear system have been established and its solution has been derived in [14,4,7]. The reachability and controllability of singular systems have been considered in many papers [1,4]. Necessary and sufficient conditions for reachability and controllability of standard positive linear systems have been established in [3,15].

Positive singular discrete-time linear systems have been analysed in [4,7]. The relationship between positive systems and electrical circuits has been considered in [6]. In this paper a new class of weakly positive singular systems will be introduced. Necessary and sufficient conditions will be established under which a weakly positive singular continuous-time system can be transformed by the strict equivalence to a positive system. It will be shown that linear electrical circuits consisting of resistances, inductances (capacitances) and source voltages are examples of positive singular continuous-time linear systems.

2 Preliminaries

Let $\mathbb{R}^{n \times m}$ be the set of $n \times m$ real matrices and $\mathbb{R}^n := \mathbb{R}^{n \times 1}$. Consider the singular continuous-time linear system

$$Ex = Ax + Bu, \quad x(0) = x_o$$

(1a)
\[
y = Cx + Du
\]
where \( x \in \mathbb{R}^n \) is the semistate vector, \( u \in \mathbb{R}^m \) is the input vector, \( y \in \mathbb{R}^p \) is the output vector and \( E \in \mathbb{R}^{n \times n} \), \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{p \times n} \), \( D \in \mathbb{R}^{p \times m} \) with \( E \) possibly singular.

**Definition 1.** The system (1) is called standard if and only if \( E = I_n \) (the identity matrix)

**Definition 2.** The system (1) is called regular if and only if

\[
\det[E s - A] \neq 0 \quad \text{for some} \quad s \in \mathbb{C} \quad \text{(the field of complex numbers)}
\]

Let \( \mathbb{R}_+^n \) be the set of \( n \)-dimensional real vectors with nonnegative components.

**Definition 3.** The system (1) is called positive if and only if for all \( x_0 \in \mathbb{R}_+^n \) and \( u(t) = u \in \mathbb{R}_+^m \), \( t \geq 0 \) we have \( x(t) = x \in \mathbb{R}_+^n \), \( t \geq 0 \) and \( y(t) = y \in \mathbb{R}_+^p \), \( t \geq 0 \).

**Definition 4.** A matrix \( A \in \mathbb{R}^{n \times n} \) is called the Metzler matrix if all its off-diagonal entries are nonnegative.

It is easy to show [6] that \( e^{At} \in \mathbb{R}_+^{n \times n} \) if and only if \( A \) is a Metzler matrix. It is well-known [6] that the standard system (1) (with \( E = I_n \)) is positive if and only if \( A \) is a Metzler matrix and \( B \in \mathbb{R}_+^{n \times m} \), \( C \in \mathbb{R}_+^{p \times n} \), \( D \in \mathbb{R}_+^{p \times m} \). Taking into account this fact the following definition of weakly positive continuous-time linear systems is introduced.

**Definition 5.** The system (1) is called weakly positive if and only if \( A \) is a Metzler matrix and \( E \in \mathbb{R}_+^{n \times n} \), \( B \in \mathbb{R}_+^{n \times m} \), \( C \in \mathbb{R}_+^{p \times n} \), \( D \in \mathbb{R}_+^{p \times m} \).

If the system (1) is regular then [14]

\[
[Es - A]^{-1} = \sum_{i=-\mu}^{\infty} \Phi_i s^{-(i+1)}
\]

where \( \mu = \text{rank} \ E - \deg(\det[Es - A]) + 1 \) is the index of nilpotence and \( \Phi_i \) is the fundamental matrix defined by