7. \( H_\infty \) Loop-Shaping

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7.1 Preliminaries

The gain of a matrix \( A \in \mathbb{R}^{n \times m} \) can be defined as \( \frac{\|Ax\|_2}{\|x\|_2} \) where \( x \in \mathbb{R}^m \) is an input vector and \( \|.\|_2 \) denotes the Euclidean 2-norm. It can easily be deduced, after a few calculations, that the gain of \( A \) will depend on the direction of the input vector \( x \). To see this we define the singular value decomposition (SVD) of a matrix (see pp. 32-35 in [266]). For example the SVD of a matrix \( A \in \mathbb{R}^{2 \times 2} \) is

\[
A = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma(A) & 0 \\ 0 & \sigma(A) \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix},
\]

where \( u_1, u_2, v_1, v_2 \in \mathbb{R}^2 \) and \( \begin{bmatrix} u_1 & u_2 \end{bmatrix} \) and \( \begin{bmatrix} v_1 & v_2 \end{bmatrix} \) are unitary matrices (see p. 19 in [266]). \( \sigma(A) \) denotes the maximum singular value of \( A \) and \( \sigma(A) \) the minimum singular value of \( A \). Therefore, as

\[
\begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

if \( x = v_1 \) then the gain of \( A \) will be \( \sigma(A) \) with \( Ax \) in the direction of \( u_1 \). Similarly, for \( x = v_2 \) the gain will be \( \sigma(A) \) with \( Ax \) in the direction of \( u_2 \).

Define the condition number of \( A \) as

\[
\kappa(A) = \frac{\sigma(A)}{\sigma(A)}.
\]

Hence an ill-conditioned matrix, i.e. a matrix with a high condition number, has significantly different gains in different directions. A “round matrix” is a matrix with a condition number near unity.

Assume that we are given a linear time invariant (LTI) stable system \( G \) with \( m \) inputs and \( p \) outputs, i.e. \( G(j\omega) \in \mathbb{C}^{p \times m} \). All inputs \( u(t) \) into the system are assumed to have finite energy, i.e. a finite 2-norm. We define the 2-norm of a signal \( u(t) \) as

\[
\|u\|_2 = \left( \int_{-\infty}^{+\infty} u^* u dt \right)^{\frac{1}{2}}.
\]
It is easy to prove (see pp. 18-25 in [61]) that if \( \|u\|_2 \leq 1 \) the energy of the output \( y(t) \) will be bounded by

\[
\sup \{ \|y\|_2 : \|u\|_2 \leq 1 \} = \|G\|_\infty
\]

where the \( \infty \)-norm of \( G \) is defined as

\[
\|G\|_\infty = \sup_{\omega} |G(j\omega)|.
\]

Therefore \( \|G\|_\infty \) denotes the maximum gain of \( G \) over all frequencies and all input directions. Consequently, if for example we wanted good disturbance rejection at the output of a plant we would try to minimise the \( \infty \)-norm of the output sensitivity. The \( \infty \)-norm can be a conservative measure if we are not interested in all input directions. This motivates the use of the structured singular value \( \mu \) (see pp. 271-300 in [266]).

If \( P \) is a given plant model, then \( P = M^{-1}N \) is a normalised left coprime factorisation of \( P \) where \( M \) and \( N \) are stable rational transfer matrices satisfying the normalisation constraint

\[
\hat{N}N^* + \hat{M}M^* = I.
\]

The motivation for using \( H_\infty \) techniques to design robust controllers is provided by the small gain theorem (see pp. 217-221 in [266]). From the small gain theorem it can be deduced that by minimising the \( \infty \)-norm of a stable transfer matrix we maximise the size, in an \( \infty \)-norm sense, of the unstructured perturbation to which the system remains stable. Hence a typical \( H_\infty \) control problem would be to minimise the \( \infty \)-norm of a transfer matrix, called the generalised plant, over all stabilising controllers. This optimisation problem has an exact solution [63]. The transfer matrix we choose to look at depends on the type of uncertainty present in the plant and the performance specifications.

The first step of a typical \( H_\infty \) design procedure would be to decide on the type of uncertainty to be used (see table on p. 227 in [266]). This is difficult and requires good knowledge of the plant. Normalised coprime factor uncertainty is the most general type of unstructured uncertainty (see pp. 418-419 in [96]). The second step would be to choose frequency dependent weights according to performance specifications and solve the optimisation problem (see pp. 213-245 in [266]). Some well studied \( H_\infty \) controller design techniques are \( H_\infty \) loop-shaping, the S/KS design procedure (see Chapter 6) and \( \mu \)-synthesis (see Chapter 8).

### 7.2 Overview of the Design Procedure

The \( H_\infty \) loop-shaping design procedure [164] is described below:

1. Shape \( G \) open loop with frequency dependent weights \( W_1 \) and \( W_2 \) according to closed loop objectives. The weighted plant, \( G_* = W_2GW_1 \), is depicted in Figure 7.1.