1. Introduction

As with the dynamics of galaxies and cosmological simulations, which are described in the papers by van Albada and Efstathiou in this volume, we are concerned essentially with the gravitational interaction of \( N \) point masses. Where our interest differs from that in galactic dynamics is that the effect of fluctuations, or graininess, in the gravitational field is crucial, and it differs from cosmological simulations in that clusters can, up to a point, be considered in isolation from the rest of the universe.

Of the three essentially distinct models for the dynamics of open and globular star clusters, the first (the \( N \)-body model) is equivalent to a Liouville equation in \( 6N \)-dimensional phase-space, i.e.

\[
\frac{\partial f}{\partial t} + \sum_{i=1}^{N} (v \cdot \frac{\partial f}{\partial r_{i}} - \frac{\partial \phi}{\partial r_{i}} \cdot \frac{\partial f}{\partial v_{i}}) = 0, 
\]

where \( f \) is the distribution of the positions, \( r_{i} \), and velocities, \( v_{i} \), of the \( N \) particles, and \( \phi_{i} \) is the gravitational potential at \( r_{i} \). The second model (the Fokker-Planck model) is obtained by integrating over all but one of the bodies, and by approximating the effect of two-body correlations in a certain way. This yields an evolution equation for the one-body distribution \( f(r, v) \), and after some further approximations gives rise to eqs. (3.1-2) below. The third and last model (the fluid model) can be justified by taking moments of the Fokker-Planck equation, i.e. multiplying by powers of \( v \) and integrating over \( v \). Again some further simplifications are needed, and lead to eqs.(4.1-4).

This discussion should suggest that the fluid model is the simplest to work with, but that the \( N \)-body model is the one most free of approximations. This inference is broadly correct, though the more simplified models are harder to modify if extra dynamical processes (such as mass loss) are to be incorporated. But experience shows that all three models have played important parts in the development of the dynamical theory of star clusters, and each has different advantages over the others.

2. The \( N \)-body model

2.1 Equations

The simplest \( N \)-body model is given by the equations

\[
\ddot{r}_{i} = -G \sum_{j=1, j \neq i}^{N} \frac{m_{j} r_{i} - r_{j}}{|r_{i} - r_{j}|^{3}} (i = 1, 2, \cdots, N),
\]

where \( r_{i} \) is the position vector of the \( i \)th body, whose mass is \( m_{i} \). Several additional processes can be included easily by means of modifications to the right-hand side, and are discussed in §2.5 below. The system (2.1) is equivalent to 6\( N \) first-order ordinary differential equations.
2.2 Exact results

It can be very useful to have exact results for the purpose of testing and checking numerical computations. Exact solutions of (2.1) are only useful for the case $N = 2$, but for general $N$ there are ten classical integrals, i.e. the position and velocity of the centre of mass (6 integrals), the total angular momentum (3 integrals) and the total energy. Experience shows that the last of these is the most useful for checking the accuracy of numerical computations, although there is no reason for supposing that accurate energy conservation is sufficient to ensure accuracy of the solutions, for reasons implicit in §2.3 below.

Another exact and useful property of eqs. (2.1) is their time-reversal invariance. As a practical check this is more time-consuming than the energy check, but it is also thought to be far more stringent, except, of course, for those integration algorithms which automatically ensure time-reversal invariance.

Finally, there is a large sequence of invariants called Poincaré invariants (see, for example, Goldstein 1980 or Arnold 1978), which exist because of the Hamiltonian form of eqs. (2.1). Unfortunately they cannot be used as checks on individual solutions, but are a result about neighbouring solutions of the equations. The simplest invariant is the 2-form $dp \wedge dq$, where $p, q$ are (respectively) the vectors of momenta and positions of the $N$ particles; thus each vector has $3N$ components. The form can be written as

$$dp \wedge dq = \sum_{i=1}^{2N} \left| \begin{array}{c} dp_i^1 \\ dq_i^1 \\ dq_i^2 \end{array} \right|,$$

where $dp_i^1$ is the $i$th momentum-component of the vector joining the solution $(q, p)$ to the neighbouring solution $(q + dq_i^1, p + dp_i^1)$, and similarly for the quantities $dq_i^1, dp_i^2, dq_i^2$. Thus the invariance of this 2-form could be tested by integrating three neighbouring solutions, or via the variational equations (Miller 1971).

The other Poincaré invariants are powers (in the sense of exterior calculus) of the above two-form, and progressively become more expensive to compute. But the last of them is of importance, because it is the volume element in the $6N$-dimensional phase space of the $N$-body problem. Its invariance is of importance for the statistical behaviour of the system. Since it is statistical results that are usually wanted in this problem (§2.3), it is of importance to conserve phase-space volume sufficiently accurately, and since this invariant is related to the simpler invariant (2.2), it would be of interest to check the invariance of $dp \wedge dq$ in some test calculations. Another approach to this would be to devise integration algorithms which automatically conserve the Poincaré invariants. Perhaps this could be done by working in terms of the generating function of the canonical transformation from positions and momenta at time $t$ to values at $t + \Delta t$ (Goldstein 1980).

2.3 Nature of the solutions

It has been shown (Miller 1964, 1971) that small $N$-body systems ($4 \leq N \leq 32$) are unstable, on the relatively short time scale of about one-fifth of a crossing time for $N = 32$. (Miller gives the $N$-dependence as roughly proportional to $N^{-4/3} \tau_{cr}$.) Thus if $\Delta(t)$ is the error (distance in $6N$-dimensional phase space) resulting from an initial error $\Delta(0)$, we have $\Delta(t) \approx \Delta(0) \exp(5t/\tau_{cr})$ for $N = 32$, provided that the errors remain small. Since the half-life of an 8-body system is roughly $35\tau_{cr}$ (Casertano 1985), and the half-life is expected on theoretical grounds to grow roughly as $N\tau_{cr}$, the growth of the error in a 32-body system during its half-life is of order $10^{2000}$. Thus hundreds of significant figures, and an integration algorithm to match, would be needed.

The pragmatic approach to this difficulty is to argue that, while the detailed results of a numerical integration cannot be reliable, the statistical results are. There is little more than common sense to justify this. There are theorems ('shadowing' lemmas, cf. Guckenheimer & Holmes 1983) which assert that, for certain kinds of dynamical system, there is an exact orbit