Abstract. An optimal shape design problem of an elastic body described by system of two nonlinear elliptic equations of the fourth order is considered. The problem is to find the boundary of the domain occupied by the body in such a way that the cost functional approximating the stiffness of the system in the equilibrium state is minimized. It is assumed that the volume of the body is constant. Moreover the function describing the boundary of the domain and its gradient are bounded. Necessary optimality condition for this problem is formulated using material derivative method.

1. Introduction

This paper is concerned with a shape optimization problem for an elastic nonlinear plate. The equilibrium state of this plate is described by system of two nonlinear, coupled elliptic equations of the fourth order. These equations are called von Karman equations. Existence of solutions and convergence of the finite dimensional approximation for von Karman systems was investigated in [6, 12].

The optimization problem considered in this paper consists in minimizing the cost functional approximating the plate stiffness with respect to the domain occupied by the plate. Moreover the function describing the boundary of the domain and its gradient are bounded. The cost functional is nondifferentiable. In literature [2, 9] such problems where the domain occupied by the body is variable subject to optimization are called shape optimization problems.

In literature [2, 9, 14, 18] most authors have considered such optimization problems for linear elliptic problems. A few authors only have studied shape optimization problems for
nonlinear elliptic equations [5, 8, 15, 16] or elliptic variational inequalities [3, 10, 17].

The goal of this paper is to investigate this shape optimization problem for strongly nonlinear von Karman system. We determine the directional derivative of the cost functional of this shape optimization problem with respect to the domain occupied by the plate. In order to do it we employ developed by Zolesio [19] speed method. We shall formulate necessary optimality condition for this nondifferentiable optimization problem.

2. Formulation of the problem

Consider an elastic plate occupying in the plane \(Ox_1x_2\) domain \(\Omega = \Omega(v)\) given by:

\[
(\Omega(v)) = \{ x = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < v(x_2), 0 < x_2 < 1 \} \tag{2.1}
\]

Lipschitz continuous function \(v(x_2)\) satisfies:

\[
v \in U = \{ z \in C^{0,1}(0,1) : 0 < a_1 \leq v(x_2) \leq \bar{v}_2 \}
\]

where \(a_1\), \(\bar{v}_2\) are given positive constants.

We shall assume that the domain \(\Omega(v)\) is simply connected. The boundary \(\Gamma\) of the domain \(\Omega(v)\) is Lipschitz continuous and consists of two parts \(\Gamma_1\) and \(\Gamma_2\) such that \(\Gamma = \Gamma_1 \cup \Gamma_2\), \(\Gamma_1 \cap \Gamma_2 = \emptyset\), \(\Gamma_2 = \Gamma_2(v)\) where \(\Gamma_2(v) = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 = v(x_2), 0 < x_2 < 1 \}\).

We shall consider nonlinear model of an elastic plate. Let us denote by \(w = w(v(x))\) the deflection of the plate and by \(f = f(v(x))\) the Airy's stress function [6, 11]. Let \(d\) be a perpendicular force bending the plate.

Before we formulate the problem let us introduce the necessary notation. Sobolev spaces \(H^1_0(\Omega), H^2(\Omega), H^2_0(\Omega)\) are defined as follows [1]:

\[
H^1_0(\Omega) = \{ z \in L^2(\Omega) : z_1 \in L^2(\Omega), z = 0 \text{ on } \Gamma \}
\]

\[
H^2(\Omega) = \{ z \in L^2(\Omega) : z_1, z_{1j} \in L^2(\Omega) \}
\]

\[
H^2_0(\Omega) = \{ z \in H^2(\Omega) : z = \partial z/\partial n = 0 \text{ on } \Gamma \}
\]

\[
z_1 = \partial z/\partial x_1, \quad z_{1j} = \partial z/\partial x_1 \partial x_j, \quad i, j = 1, 2
\]

\(\partial z/\partial n\) denotes the outward normal derivative of \(z\) on \(\Gamma\), \(n = (n_1, n_2)\) is the unit outward normal versor to the boundary \(\Gamma\).

Let us introduce the forms: \(a(\cdot, \cdot) : H^2_0(\Omega) \times H^2_0(\Omega) \rightarrow \mathbb{R}\),