On formulating a non-linear numerical model in three dimensions for tides and storm surges

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Introduction

This paper is the latest of a series of contributions on the numerical solution of the three-dimensional hydrodynamic equations for tides and storm surges. The foundation of all the work is a new mathematical method in which the horizontal components of current are expanded in terms of eigenfunctions through the depth. Coefficients of the expansion, varying in the horizontal and with time, are determined from a two-dimensional numerical time-stepping procedure. Thereby, the developing three-dimensional current structure is computed, also the changing pattern of sea-surface elevation.

In the first place, the method was tested in a simple situation by determining the motion produced in a closed rectangular sea basin by a uniform wind suddenly imposed (Heaps 1972). Investigations of wind effects in the Irish Sea followed, determining the dynamic response of that sea to uniform southerly and westerly wind fields. In both these cases the three-dimensional motion was computed and, after transient oscillations had been damped down by friction, the steady-state circulation maintained by the wind was found (Heaps 1973, 1974). Essentially the same three-dimensional numerical model which formed the basis of the preceding calculations was then used to compute storm surges in the Irish Sea for a specific period, namely 10 to 18 January 1965. As a result, measurements of actual sea-surface elevation and certain cross-sectional flows were satisfactorily reproduced. This constituted the first verification of the new modelling technique, employing observational data (Heaps and Jones 1975).

Assumptions have been made as follows:

(i) the water is homogeneous,

(ii) pressure obeys the hydrostatic law,

(iii) stresses associated with horizontal shears are ignored,
(iv) the coefficient of vertical eddy viscosity is regarded as uniform with respect to the depth coordinate,

(v) a slip condition holds at the sea bed, with linear bottom friction,

(vi) non-linear effects are omitted.

The purpose of the present work is to suggest how the three-dimensional numerical modelling, developed so far as outlined above, might be extended to include non-linear terms in the hydrodynamic equations, taking into account a quadratic rather than a linear law of bottom friction. This is an important step since, for tide alone and for combinations of tide and surge, non-linear effects frequently have a significant influence on the motion, particularly in shallow seas. Clarke (1974) has already discussed the inclusion of quadratic friction employing a somewhat different approach to that given here.

The proposed extension to take account of non-linearities is mapped out here in mathematical style without application. The formulation is exploratory and tentative, being ultimately dependent for its success on satisfactory performance in actual computations.

**Basic equations**

Assuming that the water is homogeneous, that pressure obeys the hydrostatic law, and that shear stress in horizontal planes may be ignored, from hydrodynamics:

\[
\frac{\partial \omega}{\partial t} + \frac{\partial}{\partial x} \int_0^L u \, dz + \frac{\partial}{\partial y} \int_0^L v \, dz = 0 ,
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial z} - \gamma v' = - \frac{3}{2} \frac{\partial}{\partial x} (\xi - \xi') + \frac{\partial}{\partial z} \left( N \frac{\partial \xi}{\partial z} \right) ,
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial z} + \gamma u' = - \frac{3}{2} \frac{\partial}{\partial y} (\xi - \xi') + \frac{\partial}{\partial z} \left( N \frac{\partial \xi}{\partial z} \right) ,
\]

\[
\omega = \frac{\partial}{\partial x} \int_0^L u \, dz + \frac{\partial}{\partial y} \int_0^L v \, dz ,
\]