Mathematical Programming Study 19 (1982) 77–100.
North-Holland Publishing Company

CHARACTERIZATIONS OF OPTIMALITY WITHOUT CONSTRAINT QUALIFICATION FOR THE ABSTRACT CONVEX PROGRAM

J.M. BORWEIN* and H. WOLKOWICZ**

Department of Mathematics, Dalhousie University, Halifax, N.S. B3H 4H8, Canada

Received 21 June 1979
Revised manuscript received 15 July 1980

We consider the general abstract convex program

(P) $\qquad$ minimize $f(x)$, subject to $g(x) \in -S$,

where f is an extended convex functional on X, $g: X \to Y$ is S-convex, S is a closed convex cone and X and Y are topological linear spaces. We present primal and dual characterizations for (P). These characterizations are derived by reducing the problem to a standard Lagrange multiplier problem. Examples given include operator constrained problems as well as semi-infinite programming problems.

Key words: Cone-convex, Locally Convex Topological Vector Space, Optimality Conditions, Subdifferential, Directional Derivative, Faithfully Convex, Lagrange Multipliers, Slater's Condition, Semi-infinite Programs.

1. Introduction

We consider the abstract convex program

$$\text{(P)} \qquad \begin{array}{ll} \text{minimize} & f(x), \\ \text{subject to} & g(x) \in -S, \end{array}$$

where $f: X \to R \cup \{\infty\}$ is convex, $g: X \to Y$, X and Y are real locally convex (Hausdorff) spaces, S is a closed convex cone and g is S-convex. Recently Ben-Israel et al. [3] have presented a characterization of optimality, without a constraint qualification, in the case that the constraint g is given by the finite number of real-valued constraints $g^k(x) \le 0$, $k = 1, \dots, m$. They relied heavily on the convexity properties of the functions and, in particular, they used the cone of directions of constancy of the 'equality' constraints (see e.g. Abrams and Kerzner [1]).

Many people have considered optimality criteria for the abstract program (P) (see e.g. Holmes [14], Kurcyusz [15], Zowe and Kurcyusz [25], Luenberger [16] and Neustadt [17] and the references therein). Their criteria required a constraint

* Research partially supported by NRC A4493. Presently at the Department of Mathematics, Carnegie-Mellon University, Pittsburgh, PA, U.S.A.

** Research partially supported by NSERC A 3388. Presently at the Department of Mathematics, The University of Alberta, Edmonton, Alta., Canada.

qualification. Craven and Zlobec [8] have extended the work in [3] in order to get a characterization of optimality for (P) that does not require any constraint qualification. They, however, required the following assumptions: (i) the cone S has nonempty interior (this automatically guarantees that the dual cone has a compact base; thus, when they choose a compact subset of the dual cone, they might as well choose a base); (ii) the feasible set contains a relative radial (core) point; (iii) continuity and differentiability properties; and (iv) the infimum is attained. They also used the cone of constancy of the 'equality' constraints in a redundant manner.

In this paper we give a characterization which avoids the above-mentioned assumptions. Rather than attempt to extend the results in [3], our results are based on reducing (P) so as to be able to apply the 'Standard Lagrange Multiplier theorem'. We then give two classes of optimality criteria.

The organization of the paper is as follows. Section 2 presents several preliminary definitions and results. In particular, Lemma 2.2 finds a 'Slater point' for any compact subset of the 'nonequality' constraints; Lemma 2.3 and Corollary 2.1 characterize the existence of a compact base and Slater's condition; and Lemma 2.5 gives a dual relationship between the cone of subgradients and the linearizing cone.

Section 3 presents the 'Standard Lagrange Multiplier theorem' for program (P) with the added constraint $x \in \Omega$, where Ω is a convex subset of X (see Theorem 3.1). Several different types of optimality criteria are given. These criteria use directional derivatives, subgradients and the Lagrangian function.

Section 4 presents the complete characterization of optimality without any constraint qualification (see Theorem 4.1 and Corollary 4.1). These results are derived using the results in Section 3. Several corollaries are also given, including the result which leads to the BBZ conditions [3], (see Corollary 4.2 and the following remarks).

Section 5 contains several examples and applications which illustrate the theory developed in the first four sections.

2. Preliminaries

In this section we present some preliminary definitions and results needed in the sequel. We consider the *convex program*

$$\text{(P)} \qquad \begin{array}{l} \text{minimize} \quad f(x), \\ \text{subject to } g(x) \in -S, \end{array}$$

where $f: X \to R \cup \{\infty\}$ is convex, $g: X \to Y$, X and Y are real locally convex (Hausdorff) spaces, S is a closed convex cone in Y and g is S-convex, i.e. $S + S \subset S$, $\lambda S \subset S$ for all positive λ and for all x, y in X,