Axiomatizing complex algebras by games

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Abstract. Given a variety $V$, we provide an axiomatization $\Phi(V)$ of the class $\text{SCmV}$ of complex algebras of algebras in $V$. $\Phi(V)$ can be obtained effectively from the axiomatization of $V$; in fact, if this axiomatization is recursively enumerable, then $\Phi(V)$ is recursive.

1. Introduction

The construction of complexes of structures is a standard procedure in mathematics. Probably the oldest and best known example is found in group theory: given a group, consider the algebra whose carrier is the power set of the group elements and whose operations are the power lifts of the group operations, for instance,

$$X \circ Y = \{ x \circ y \mid x \in X, y \in Y \}.$$

In lattice theory it is well known that the set of ideals of a distributive lattice $L$ again forms a lattice, of which the meet and join coincide with the lifted meet and join operations of $L$, respectively:

$$I_1 \lor I_2 = \{ a_1 \lor a_2 \mid a_1 \in I_1, a_2 \in I_2 \},$$
$$I_1 \land I_2 = \{ a_1 \land a_2 \mid a_1 \in I_1, a_2 \in I_2 \}.$$

And as a last example we mention formal language theory, where we may see the product of two languages as the lift of word concatenation:

$$L_1; L_2 = \{ w_1 w_2 \mid w_1 \in L_1, w_2 \in L_2 \}.$$

Obviously, this construction can be carried out for an arbitrary operation, giving rise to the power algebra of an algebra (formal definitions are found in the next section). Since
the universe of such an algebra is a power set algebra, it is natural to include the Boolean operations into the similarity type; thus we obtain the full complex algebra $\mathcal{L}^+$ of an algebra $\mathcal{L}$. If instead of all subsets of $\mathcal{L}$ we take as carrier of the algebra some non-empty collection of subsets of $\mathcal{L}$ that is closed under the Boolean operations and under the lifted operations, we get an arbitrary complex algebra over $\mathcal{L}$; formulated more concisely, a complex algebra over $\mathcal{L}$ is any subalgebra of $\mathcal{L}^+$.

For some notation, given a class $\mathcal{K}$ of algebras, we denote the class of full complex algebras over (algebras in) $\mathcal{K}$ by $\mathcal{Cm}\mathcal{K}$; $\mathcal{SCm}\mathcal{K}$ denotes the class of isomorphism types of complex algebras over $\mathcal{K}$, and $\mathcal{Var}_\mathcal{K}$, the variety generated by $\mathcal{Cm}\mathcal{K}$.

The construction gives rise to various questions of a universal algebraic nature, for instance concerning the relation between a class $\mathcal{K}$ of algebras and the class $\mathcal{SCm}\mathcal{K}$ of associated complex algebras. For a survey of known results and references to the literature we refer the reader to Brink [1] and Goldblatt [4] (the second paper takes a more general perspective, considering complex algebras of arbitrary relational structures).

In this paper, we are interested in finding an axiomatization of the class of complex algebras of a given variety $\mathcal{V}$. It seems that in the general case, not much is known. There are some known results relating the validity of an equation in an algebra to its validity in the power algebra. For instance, a result by Gautam [3] states that the validity of an equation is preserved under moving to the power algebra if and only if every variable in the equation occurs exactly once on each side of the equation. This makes it improbable that an equational axiomatization of a variety $\mathcal{V}$ will be of direct use in finding an axiomatization of $\mathcal{SCm}\mathcal{V}$.

Recently, Goranko and Vakarelov [5] have given complete axiomatizations of the modal logic of various classes of relational structures, including varieties of algebras. Translated into algebraic terms, their result yields a derivation system for the set $\text{Equ}(\mathcal{Cm}\mathcal{V})$ of equations valid in the class $\mathcal{Cm}\mathcal{V}$ for an arbitrary variety $\mathcal{V}$. Their result crucially involves the extension of the lifted algebraic language with a so-called difference operator, and an extension of the derivation system with a non-structural derivation rule. However, for some varieties $\mathcal{V}$, including groups and (thus) Boolean algebras, this difference operator is term-definable over the class $\mathcal{Cm}\mathcal{V}$. Hence, for such a variety $\mathcal{V}$, the result of Goranko and Vakarelov provides a derivation system for the equational theory $\text{Equ}(\mathcal{Cm}\mathcal{V})$ within the language of the complex algebras — but since this system has a non-structural rule, it is not an equational axiomatization in the traditional sense, or an equational characterization of the variety $\mathcal{Var}_\mathcal{V}$. Independently, Venema [17] obtained the same result for the case of groups.

In the case of groups, some other results are known. Complex algebras of groups appear in the literature on algebraic logic as group relation algebras, GRAs. Tarski [15] showed that GRAs is axiomatizable by a set of equations over the class of integral relation algebras, while McKenzie [13] proved that no finite axiomatization of GRAs can be found. McKenzie [13, p. 282] writes:

"It would certainly be of interest to have a reasonably elegant system of first-order axioms characterizing [GRA]."