SUBSPACES OF $c_0(\mathbb{N})$ AND LIPSCHITZ ISOMORPHISMS

G. GODEFROY, N. KALTON AND G. LANCIEN

Abstract

We show that the class of subspaces of $c_0(\mathbb{N})$ is stable under Lipschitz isomorphisms. The main corollary is that any Banach space which is Lipschitz-isomorphic to $c_0(\mathbb{N})$ is linearly isomorphic to $c_0(\mathbb{N})$. The proof relies in part on an isomorphic characterization of subspaces of $c_0(\mathbb{N})$ as separable spaces having an equivalent norm such that the weak-star and norm topologies quantitatively agree on the dual unit sphere. Estimates on the Banach–Mazur distances are provided when the Lipschitz constants of the isomorphisms are small. The quite different non-separable theory is also investigated.

1 Introduction

Banach spaces are usually considered within the category of topological vector spaces, and isomorphisms between them are assumed to be continuous and linear. It is however natural to study them from different points of view, e.g. as infinite dimensional smooth manifolds, metric spaces or uniform spaces, and to investigate whether this actually leads to different isomorphism classes. We refer to [JoLS] and references therein for recent results and description of this field. Some simply stated questions turn out to be hard to answer: for instance, no examples are known of separable Banach spaces $X$ and $Y$ which are Lipschitz isomorphic but not linearly isomorphic. It is not even known if this could occur when $X$ is isomorphic to $l_1$. The main result of this work is that any separable space which is Lipschitz isomorphic to $c_0(\mathbb{N})$ is linearly isomorphic to $c_0(\mathbb{N})$. Showing it will require the use of various tools from non-linear functional analysis, such as the Gorelik principle. New linear results on subspaces of $c_0(\mathbb{N})$ will also be needed.

We now turn to a detailed description of our results. Section 2 contains the main theorems of our article (Theorems 2.1 and 2.2), which contribute to the classification of separable Banach spaces under Lipschitz isomorphisms. These results are non-linear. However, their proof requires linear
tools such as Theorem 2.4 which provides a characterization of linear subspaces of $c_0(N)$ in terms of existence of equivalent norms with a property of asymptotic uniform smoothness. This technical property is easier to handle through the dual norm, which is such that the weak* and norm topologies agree quantitatively on the sphere (see Definition 2.3). The main topological argument we need is Gorelik’s principle (Proposition 2.7) which is combined with a renorming technique and with Theorem 2.4 for showing (Theorem 2.1) that the class of subspaces of $c_0(N)$ is stable under Lipschitz-isomorphisms. It follows (Theorem 2.2) that a Banach space is isomorphic to $c_0(N)$ as soon as it is Lipschitz-isomorphic to it. The renorming technique is somewhat similar to “maximal rate of change” arguments which are used for differentiating Lipschitz functions (see [P]).

We subsequently investigate extensions of the separable isomorphic results of section 2 in two directions: what can be said when the Lipschitz constants of the Lipschitz isomorphisms are small? What happens in the non-separable case? These questions are answered in the last three sections. For reaching the answers, we have to use specific tools, since the proofs are not straightforward extensions of those from section 2.

Section 3 deals with quantitative versions of Theorem 2.2. These statements are “nearly isometric” analogues, in the case of $c_0(N)$, of Mazur’s theorem which states that two isometric Banach spaces are linearly isometric. Indeed we show that a Banach space $X$ is close to $c_0(N)$ in Banach–Mazur distance if there is a Lipschitz-isomorphism $U$ between $X$ and $c_0(N)$ such that the Lipschitz constants of $U$ and $U^{-1}$ are close to 1 (Propositions 3.2 and 3.4). Proposition 3.2 relies on an examination of the proof of Gorelik’s principle in the case of $c_0(N)$ and on an unpublished result of M. Zippin ([Z3]), while Proposition 3.4 uses the concept of a $K$-space from [KaR].

The non-separable theory is studied in sections 4 and 5. It is shown in [JoLS, Theorem 6.1] that if $1 < p < \infty$, any Banach space which is uniformly homeomorphic (in particular, Lipschitz isomorphic) to $l^p(\Gamma)$ is linearly isomorphic to it, for any set $\Gamma$. But in the case of $c_0(\Gamma)$ (i.e. in the case $p = \infty$), this situation happens to be quite different. Indeed there are spaces which are Lipschitz isomorphic to $c_0(\Gamma)$ with $\Gamma$ uncountable but not linearly isomorphic to a subspace of that space (see [DGZ2] and Examples 4.9). The gist of the last two sections is that the separable theory extends to the class of weakly compactly generated spaces (that is, to spaces $X$ which contain a weakly compact subset which spans a dense linear subspace) but not further. Section 4 is devoted to characterizing