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PARTIALLY ORDERED GROUPS AND GEOMETRY OF CONTACT TRANSFORMATIONS

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Dedicated to D.B. Fuchs on the occasion of his 60th birthday

Abstract

We prove that, for a class of contact manifolds, the universal cover of the group of contact diffeomorphisms carries a natural partial order. It leads to a new viewpoint on geometry and dynamics of contactomorphisms. It gives rise to invariants of contactomorphisms which generalize the classical notion of the rotation number. Our approach is based on tools of Symplectic Topology.

1 Introduction and Main Results

1.1 Partially ordered groups. Let $\mathcal{D}$ be a group. A subset $C \subset \mathcal{D}$ is called a *normal cone* if

(i) $f \in C, g \in C \Rightarrow fg \in C$
(ii) $f \in C, h \in \mathcal{D} \Rightarrow hfh^{-1} \in C$
(iii) $1 \in C$

Given a normal cone $C \subset \mathcal{D}$, one defines a relation $f \geq g$ on $\mathcal{D}$ by

$$f \geq g \text{ if } fg^{-1} \in C\,.$$

Clearly this relation is reflexive ($f \geq f$) and transitive

$$(f \geq g, g \geq h \Rightarrow f \geq h).$$

If it is also anti-symmetric ($f \geq g, g \geq f \Rightarrow f = g$) then it is a partial order on $\mathcal{D}$. We call it a *bi-invariant partial order* induced by C. Note that the normality of the cone C implies that if $f_1 \geq g_1$ and $f_2 \geq g_2$ then $f_1 f_2 \geq g_1 g_2$.

Let us describe now a way to extract numerical invariants from a bi-invariant partial order on $\mathcal{D}$. An element $f \in C \setminus \{1\}$ is called *a dominant* if for every $g \in \mathcal{D}$ there exists a number $p \in \mathbb{N}$ such that $f^p \geq g$. For

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a dominant f and any $g \in \mathcal{D}$ set $\gamma_k(f,g) = \inf\{p \in \mathbb{Z} \mid f^p \geq g^k\}$, where $k \in \mathbb{N}$. Note that

(i) the number $\gamma_k = \gamma_k(f,g)$ is finite, and
(ii) the limit $\gamma(f,g) = \lim_{k\to+\infty} \gamma_k/k$ exists.

Indeed, choose $q \in \mathbb{N}$ such that $f^q \geq g^{-1}$. If $f^p \geq g^k$ then $g^{-k} \geq f^{-p}$, so $f^{kq} \geq f^{-p}$ and $p \geq -kq$. Hence $\gamma_k \geq -kq$ and it is finite, which proves (i). Since $f^{\gamma_n} \geq g^n$, $f^{\gamma_m} \geq g^m$ implies $f^{\gamma_m+\gamma_n} \geq g^{m+n}$ we conclude that $\gamma_{m+n} \leq \gamma_m + \gamma_n$, i.e. the sequence γ_k is subadditive. Consider now the sequence $u_k = \gamma_k + kq$ which, as we just showed, is non-negative. Clearly it is also subadditive. This implies the existence of the limit $\lim_{k\to\infty} u_k/k$, and hence of the limit (ii).

We will call $\gamma(f,g)$ *the relative growth number* of f with respect to g. Note that the real number $\gamma(f,g)$ can be of any sign, or equal to 0.

If both f and g are dominants, then $\gamma(g,f)$ is also defined, and the following inequality holds:

$$\gamma(g,f)\gamma(f,g) \geq 1\,. \tag{1.1.A}$$

Indeed, set $\alpha_k = \gamma_k(f,g)$ and $\beta_k = \gamma_k(g,f)$. Then we have $f^{\alpha_k} \geq g^k$ and $g^{\beta_k} \geq f^k$. Hence,

$$g^{\alpha_k\beta_k} \geq f^{k\alpha_k} \geq g^{k^2}\,.$$

Since g is a dominant this implies that $\alpha_k\beta_k \geq k^2$. Dividing by k^2 both parts of this inequality and passing to the limit when $k \to +\infty$ we get the required inequality (1.1.A).

1.2 The universal cover of the group of contactomorphisms. Let (M,ξ) be a closed connected contact manifold with a *co-oriented* contact structure. Let us denote by $\Gamma(M,\xi)$ the identity component of the group of contactomorphisms of (M,ξ), and by $\theta : \mathcal{D}(M,\xi) \to \Gamma(M,\xi)$ the universal cover of $\Gamma(M,\xi)$ associated with the basepoint 1. Our starting observation is that $\mathcal{D}(M,\xi)$ *carries a natural normal cone.* Let (SM,ω) be the *symplectization* of the contact manifold (M,ξ). Let us recall that SM is the total space of a $\mathbb{R}_+$-subbundle of the cotangent bundle T^*M, which is formed by contact forms compatible with the given co-orientation of ξ. The symplectic structure ω on SM is the restriction of the canonical symplectic form $d(pdq)$ of the cotangent bundle. SM also carries a canonical conformally symplectic $\mathbb{R}_+$-action. Every contactomorphism $\varphi \in \Gamma$ uniquely lifts to a $\mathbb{R}_+$-equivariant symplectomorphism $\widetilde{\varphi}$ of SM, and conversely each $\mathbb{R}_+$-equivariant symplectomorphism of SM projects to a contactomorphism of (M,ξ). A function $F : SM \to \mathbb{R}$ is called a *contact Hamiltonian* if it is