Meromorphic Parametric Non-Integrability; the Inverse Square Potential

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Abstract

We let $H(X, Y, \alpha)$ be a Hamiltonian depending meromorphically on positions $X$, inertial momenta $Y$ and parameters $\alpha$. In Theorem 1 we give conditions for the “meromorphic parametric” non-integrability of $H$.

Theorem 2 proves the meromorphic non-integrability of the 4-body problem on a line with given masses $(1, m, m, 1)$ with $m \neq 1$, and of the 3-body problem in $\mathbb{R}^p$ with $p \geq 2$ and given masses $(1, 1, m)$, for the inverse square potential. Those are the simplest cases left open after the integrability results of Jacobi (3 bodies on a line with arbitrary masses) and Calogero-Moser ($n$ bodies on a line with equal masses).

Taking the masses as parameters and using both Theorems 1 and 2, we prove Theorem 3, which shows meromorphic parametric non-integrability results for the inverse square potential.

1. Introduction

Since the beginning of celestial mechanics, the $n$-body problem with an inverse square potential has been a subject of great interest because of its rich dynamics. In this introduction, we recall the most celebrated results in the field.

Let us consider $n$ point-masses $(m_i)_{1 \leq i \leq n}$ with a potential $1/r^2$ in a $p$-dimensional Euclidean space identified with $\mathbb{R}^p$ ($p \geq 1$). Let $\| \cdot \|$ be the Euclidean norm. Positions in configuration space are denoted by $Q = (Q_1, \ldots, Q_n)$, where $Q_i = (Q_i^1, \ldots, Q_i^p)$ is the position of the $i$th mass. The configuration space is identified with $(\mathbb{R}^p)^n \setminus \Delta$, with $\Delta = \{ Q \in (\mathbb{R}^p)^n \mid \exists (i, j), \ 1 \leq i < j \leq n \ \text{and} \ \ Q_i = Q_j \}$. The phase space is the cotangent bundle $T^* (\mathbb{R}^p)^n \setminus \Delta$ of the configuration space. It is identified with $((\mathbb{R}^p)^n \setminus \Delta) \times (\mathbb{R}^p)^n$. Its elements will be denoted by:

$(Q, P) = (Q_1, \ldots, Q_n; P_1, \ldots, P_n)$,
where $P_i = m_i \frac{dQ_i}{dt}$ is the linear momentum of the $i$th point. The “force function”, of opposite sign with respect to the potential, is equal to

$$U(Q) = \varepsilon \sum_{0 \leq i < j \leq n} \frac{m_i m_j}{|Q_i - Q_j|^2},$$

where $\varepsilon = 1$ for the attractive problem and $\varepsilon = -1$ for the repulsive problem. The function $U$ is invariant with respect to the isometries of $\mathbb{R}^p$ and the corresponding equations of motion are

\[
(S) \quad \begin{cases} 
\frac{dQ_i}{dt} = \frac{\partial \mathcal{H}}{\partial P_i} = \frac{P_i}{m_i}, & \frac{dP_i}{dt} = -\frac{\partial \mathcal{H}}{\partial Q_i} = -\frac{\partial U}{\partial Q_i}, \\
\mathcal{H}(Q, P) = \frac{1}{2} \sum_{0 \leq i < j \leq n} \frac{|P_i|^2}{m_i} - U(Q).
\end{cases}
\]

In 1687, Newton [13] had already observed that two bodies in a plane, evolving under the influence of a $1/r^4$ potential, can move along logarithmic spirals around their center of mass if and only if $k = 2$.

Since collisions are very common, it is impossible to see, from the real phase portrait, whether the $n$-body problem with inverse square potential is integrable or not. The known first integrals are the energy $\mathcal{H}$, the components of angular momentum for $p \geq 2$, and the integrals associated with the uniform linear motion of the center of mass. The problem of finding first integrals which are independent of these classical first integrals is meaningful only if we look for new integrals which are rational, or at least meromorphic. Complexification of the attractive and repulsive problems makes them equivalent for meromorphic (or rational) integrability (or non-integrability). Indeed, if $f(Q, P)$ is a first integral of the attractive problem then $f(Q, iP)$ (or its real and imaginary parts) is a first integral of the repulsive problem, and vice versa.

Around 1845, Jacobi [5] proved the existence of two time-dependent first integrals for any classical Hamiltonian system with a homogeneous potential of degree -2. By eliminating the time $t$, one obtains the time-independent integral $G$. Here below we show the construction procedure.

We denote by $I = \sum_{i=1}^n m_i |Q_i|^2$ the moment of inertia with respect to the origin. It is easy to verify that $I = 4\mathcal{H}$. (For the Newtonian potential, the analogous equality is the Lagrange-Jacobi identity $I = 4\mathcal{H} + 2U(Q)$.) Let $J = I/2 = \sum_{i=1}^n Q_i \cdot P_i$. Then $J = 2\mathcal{H}t$ and $I - 2Jt + 2\mathcal{H}t^2$ are two time-dependent first integrals. Eliminating time, we find that $G = 2I\mathcal{H} - J^2$ is a time-independent first integral. For the potential $1/r^2$, $G$ is rational in $(Q, P)$ and for $n \geq 3$ it is independent from the classical first integrals. This yields the following theorem:

**Jacobi’s Theorem.** The 3-body problem on a line with arbitrary masses and inverse square potential is completely integrable with rational first integrals.

In 1906, again for the 3-body problem with an inverse square potential but in $\mathbb{R}^3$, Banachewitz [3] proved that there exist homographic configurations (i.e.,