Meromorphic Parametric Non-Integrability; the Inverse Square Potential

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Abstract

We let \( H(X, Y, \alpha) \) be a Hamiltonian depending meromorphically on positions \( X \), inertial momenta \( Y \) and parameters \( \alpha \). In Theorem 1 we give conditions for the “meromorphic parametric” non-integrability of \( H \).

Theorem 2 proves the meromorphic non-integrability of the 4-body problem on a line with given masses \((1, m, m, 1)\) with \( m \neq 1 \), and of the 3-body problem in \( \mathbb{R}^p \) with \( p \geq 2 \) and given masses \((1, 1, m)\), for the inverse square potential. Those are the simplest cases left open after the integrability results of Jacobi (3 bodies on a line with arbitrary masses) and Calogero-Moser (\( n \) bodies on a line with equal masses).

Taking the masses as parameters and using both Theorems 1 and 2, we prove Theorem 3, which shows meromorphic parametric non-integrability results for the inverse square potential.

1. Introduction

Since the beginning of celestial mechanics, the \( n \)-body problem with an inverse square potential has been a subject of great interest because of its rich dynamics. In this introduction, we recall the most celebrated results in the field.

Let us consider \( n \) point-masses \((m_i)_{1 \leq i \leq n}\) with a potential \( 1/r^2 \) in a \( p \)-dimensional Euclidean space identified with \( \mathbb{R}^p \) (\( p \geq 1 \)). Let \( \| \cdot \| \) be the Euclidean norm. Positions in configuration space are denoted by \( Q = (Q_1, \ldots, Q_n) \), where \( Q_i = (Q_1^i, \ldots, Q_p^i) \) is the position of the \( i \)th mass. The configuration space is identified with \( (\mathbb{R}^p)^n \setminus \Delta \), with \( \Delta = \{ Q \in (\mathbb{R}^p)^n \mid \exists (i, j), \ 1 \leq i < j \leq n \text{ and } Q_i = Q_j \} \).

The phase space is the cotangent bundle \( T^* ((\mathbb{R}^p)^n \setminus \Delta) \) of the configuration space. It is identified with \( ((\mathbb{R}^p)^n \setminus \Delta) \times (\mathbb{R}^p)^n \). Its elements will be denoted by:

\[ (Q, P) = (Q_1, \ldots, Q_n; P_1, \ldots, P_n), \]
where $P_i = m_i \frac{dQ_i}{dt}$ is the linear momentum of the $i$th point. The “force function”, of opposite sign with respect to the potential, is equal to

$$U(Q) = \varepsilon \sum_{0 \leq i < j \leq n} \frac{m_im_j}{||Q_i - Q_j||^2},$$

where $\varepsilon = 1$ for the attractive problem and $\varepsilon = -1$ for the repulsive problem. The function $U$ is invariant with respect to the isometries of $\mathbb{R}^p$ and the corresponding equations of motion are

$$\left\{ \begin{array}{l}
  \text{for } 1 \leq i \leq n \quad \frac{dQ_i}{dt} = \frac{\partial H}{\partial P_i} = \frac{P_i}{m_i}, \quad \frac{dP_i}{dt} = -\frac{\partial H}{\partial Q_i} = -\frac{\partial U}{\partial Q_i}, \\
  \text{with } \quad H(Q, P) = \frac{1}{2} \sum_{0 \leq i < j \leq n} \frac{||P_i||^2}{m_i} - U(Q).
\end{array} \right.$$