On the Local Representation of G-Closure

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Communicated by G. MILTON

Abstract

We give a representation for the G-closure of the set of elliptic operators
\[ \text{div} A(\cdot, \nabla u(\cdot)) : H^1_0(p\Omega; \mathbb{R}^m) \to H^{-1}(p\Omega; \mathbb{R}^m) \]
defined by means of the Nemitskii operators
\[ A(\cdot) : \Omega \times \mathbb{R}^m \to \mathbb{R}^m, \quad A(x, \xi) = \sum_{i=1}^N \chi_{\Omega_i}(x) a_i(x, \xi) \]
where the functions \(a_i\) are strongly monotone and belong to given sets of uniformly Lipschitz functions, but where the characteristic functions \(\chi_{\Omega_i}\) satisfy standard volume restrictions.

1. Introduction

The aim of this paper is to give a complete proof for the property, well known within the mathematical community, of the local representation of the G-closure of a given family of elliptic operators. For the case of a linear elliptic equation, this property can be described in the following way.

Let \(\Omega \subset \mathbb{R}^n\) be a bounded Lipschitz domain, let \(M_j, j = 1, \ldots, r\), be pairwise disjoint bounded sets of uniformly positive definite constant symmetric \(n \times n\)-matrices \(a\) and let \(d_1, \ldots, d_r\) be non-negative numbers such that \(d_1 + \cdots + d_r = |\Omega|\), \(|\Omega|\) being the Lebesgue measure of \(\Omega\). Let \(\mathcal{M}\) be the set of all measurable on \(\Omega\) matrices \(A\) such that
\[ A(x) \in \bigcup_{j=1}^r M_j \quad \text{a.e.} \ x \in \Omega, \]
\[ \left| \{ x \in \Omega \mid A(x) \in M_j \} \right| = d_j, \quad j = 1, \ldots, r. \]

We recall (see, for instance, Zhikov et al. [13]) that a sequence \(\{A_k\} \subset \mathcal{M}\) G-converges to a matrix \(A_0\) if, for every \(f \in H^{-1}(\Omega)\) from
\[ \text{div}(A_k(x)\nabla u_k(x)) = f(x) \quad \text{in} \ \Omega, \ u_k \in H^1_0(\Omega), \ k = 0, 1, 2, \ldots, \quad (1.1) \]
it follows that
\[ u_k \rightharpoonup u_0 \quad \text{weakly in } H^1_0(\Omega) \text{ as } k \to \infty, \]
\[ A_k \nabla u_k \rightharpoonup A_0 \nabla u_0 \quad \text{weakly in } L^2(\Omega; \mathbb{R}^n) \text{ as } k \to \infty. \]

The G-convergence induces on \( \mathcal{M} \) a metric \( \rho \) and in this metric the set \( \mathcal{M} \) is precompact. The completion of \( \mathcal{M} \) in the metric \( \rho \) is said to be the G-closure of \( \mathcal{M} \) and is denoted by \( G\mathcal{M} \). It is well known (see, for instance, Zhikov et al. [13]) that the set \( G\mathcal{M} \) consists of measurable positive definite matrices and the problem now is to find a description, at least formal, of \( G\mathcal{M} \).

The common agreement is (there are many references to the unpublished theorem by Dal Maso and/or Kohn) that \( G\mathcal{M} \) consists of all measurable matrices \( A^* \) such that
\[ A^*(x) \in G_{\theta}(x) \mathcal{M} \quad \text{a.e. } x \in \Omega \]
for some “distribution function” \( \theta = (\theta_1, \ldots, \theta_r) \in L^1(\Omega; \mathbb{R}^r) \) with
\[ 0 \leq \theta_j(x) \leq 1 \quad \text{a.e. } x \in \Omega, \int \theta_j(x) \, dx = d_j, \quad j = 1, \ldots, r, \]
\[ \theta_1(x) + \cdots + \theta_r(x) = 1 \quad \text{in } \Omega. \]

The set \( G_\theta \mathcal{M}, \) for a given \( \theta = (\theta_1, \ldots, \theta_r) \in \mathbb{R}^r; 0 \leq \theta_j \leq 1, \ j = 1, \ldots, r; \ \theta_1 + \cdots + \theta_r = 1, \) is defined as the closure of the set of all positive constant symmetric \( n \times n \) matrices \( a_0 \) such that for a chosen \( a_0 \) there exists a matrix \( A_0 \) such that
\[ A_0(x) \in \bigcup_{j=1}^r M_j \quad \text{a.e. } x \in K_1, \]
\[ \left| \{ x \in K_1 \mid A_0(x) \in M_j \} \right| = \theta_j, \quad j = 1, \ldots, r, \]
and that for all \( \xi^1, \xi^2 \in \mathbb{R}^n \)
\[ \langle a_0 \xi^1, \xi^2 \rangle = \int_{K_1} \langle A_0(x)(\nabla v + \xi^1), \xi^2 \rangle \, dx, \]
where \( K_1 \subset \mathbb{R}^n \) is the unit cube and \( v \) is the solution of the equation
\[ \text{div } A_0(x)(\nabla v + \xi^1) = 0 \quad \text{in } K_1 \]
with the periodic boundary conditions.

The notion of G-convergence for second order linear elliptic operators was introduced by Spagnolo [9] in the symmetric case as the weak pointwise convergence of inverse operators, and the main properties of G-convergence, such as the compactness, the type of the G-limit operator and the locality of G-convergence, were established. After a few years, De Giorgi & Spagnolo [3] obtained the local convergence of energies that gave a representation for the G-limit matrix at its Lebesgue points and later led to the principle of periodic localization (see Zhikov et al. [13]). The first results for the case of non-symmetric matrices were obtained by Tartar [10] and Murat [6]. The problem of G-convergence of quasilinear elliptic