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Rational permutation modules for finite groups

Ian Hambleton[1], **Laurence R. Taylor**[2]

[1] Department of Mathematics & Statistics, McMaster University, Hamilton, Ontario L8S 4K1, Canada (e-mail: ian@icarus.Math.McMaster.ca)
[2] Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA (e-mail: taylor.2@nd.edu)

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A rational permutation module for a finite group G is a rational representation of the form $V \cong \mathbb{Q}X$ for some finite G set X. Let $P(G)$ denote the subring of the rational representation ring $R(G)$ spanned by the permutation modules. Alternatively, $P(G)$ is the image of the Burnside ring of G in $R(G)$. Define the functor $C(G)$ as the cokernel

$$0 \longrightarrow P(G) \longrightarrow R(G) \longrightarrow C(G) \longrightarrow 0\,.$$

By the Artin Induction theorem, $C(G)$ is a finite abelian group with exponent dividing the order of G.

Some work on this sequence has already been done. In [14] and [16], Ritter and Segal proved that $C(G) = 0$ for G a finite p–group. Serre [17, p. 104] remarked that $C(G) \neq 0$ for $G = \mathbb{Z}/3 \times Q_8$ (the direct product of a cyclic group of order 3 and a quaternion group of order 8).

Berz [2] gave a nice description of $P(G)$ for G metabelian or supersolvable. To describe the result, recall that $R(G)$ additively is a free abelian group with basis given by the irreducible rational representations of G. The subgroup $P(G)$ is generated by the induced representations $\mathrm{Ind}^G(1_H) = \mathbb{Q}[G/H]$, where H runs over the subgroups of G. If a_ϕ denotes the $\gcd$ over all H of the numbers $\langle \phi, \mathrm{Ind}^G(1_H)\rangle$, then a_ϕ divides $\langle \phi, \chi\rangle$ whenever χ is a virtual permutation representation. Let $\alpha_\phi = \frac{a_\phi}{\langle \phi,\phi\rangle}$.

Theorem: (Berz [2]) *For G metabelian or supersolvable the lattice $P(G) \subseteq R(G)$ has a basis $\alpha_\phi \cdot \phi$ where ϕ runs over the irreducible rational representations of G.*

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It follows immediately from the definition that $P(G) \subseteq \bigoplus \alpha_\phi \cdot \phi$ for any finite group G. In an earlier version of this paper we claimed that equality held for all finite groups, but Berz [2] gives a counterexample. The error lay in the assertion that the lattice defined as $\bigoplus \alpha_\phi \cdot \phi$ has good induction and restriction properties.

In §1 we review some of the foundational work of A. Dress on induction theory and observe that hyperelementary computation follows for the Mackey functors $P(G)$, $R(G)$ and $C(G)$. Since hyperelementary groups are supersolvable, Berz's result applies and this information leads in principle to further information about $C(G)$ for general groups G.

In §2, we prove that the functors, P, R and C are "detected" by the *basic* subquotients of G [9]. This leads to a different proof of the Berz equality $P(G) = \bigoplus \alpha_\phi \phi$, for G hyperelementary, and to more efficient methods for computing $C(G)$.

Each basic group B is p–hyperelementary for some prime p and each basic group has a unique irreducible faithful rational representation ρ_B. Representations have induction and restriction for quotient maps as well as subgroups. Hence they also have "push forward" and "pull back" maps for subquotients. If H is a subquotient of G, we call the map from $R(G)$ to $R(H)$ the restriction and we call the map from $R(H)$ to $R(G)$ the induction map.

Hyperelementary computation and basic detection can be combined (see §3) to give an explicit numerical criterion for an arbitrary rational representation to be a virtual permutation representation.

Theorem A: *Given a rational representation χ on G, χ is a virtual permutation representation if and only if a_{ρ_B} divides $\langle \chi, \mathrm{Ind}^G(\rho_B) \rangle$ for all basic subquotients B of G.*

In §4 we describe the basic groups and give a partial calculation of the a_{ρ_B}. In conjunction with the general theory, this leads to a short proof of the Ritter–Segal theorem in §6.

In §7 we construct examples of groups G for which $C(G)$ is arbitrarily complicated. In §8 we give some calculations of $C(G)$ and prove some vanishing results. One consequence of Corollary 8.3 is:

Theorem B: *If p is the largest prime dividing the order of G, then $C(G)$ is p–torsion free.*

To state the calculation for G nilpotent (see §§9-10) we need some notation. Let $Ch_{\mathbb{Q}}(G)$ denote the ring of rational valued *characters* of G, and recall that

(0.1) $$0 \to R(G) \to Ch_{\mathbb{Q}}(G) \to \bigoplus \mathbb{Z}/m_\phi \to 0$$

is a short exact sequence where the sum runs over the irreducible rational representations ϕ of G and m_ϕ is the Schur index of an irreducible complex