A note on conjugate gradient convergence – Part III

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Summary. In this paper we again consider the rate of convergence of the conjugate gradient method. We start with a general analysis of the conjugate gradient method for uniformly bounded solutions vectors and matrices whose eigenvalues are uniformly bounded and positive. We show that in such cases a fixed finite number of iterations of the method gives some fixed amount of improvement as the the size of the matrix tends to infinity. Then we specialize to the finite element (or finite difference) scheme for the problem \( y''(x) = g_\beta(x), \quad y(0) = y(1) = 0 \). We show that for some classes of function \( g_\beta \) we see this same effect. For other functions we show that the gain made by performing a fixed number of iterations of the method tends to zero as the size of the matrix tends to infinity.

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1. Overview

In this paper we again consider the rate of convergence of the conjugate gradient method. As in our previous papers [1, 2], we consider the problem:

\[
y''(x) = g_\beta(x), \quad y(0) = y(1) = 0.
\]
Approximating this problem using either the method of finite elements or the method of finite differences leads to the matrix equation:

\[
\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \ddots & \vdots \\
0 & -1 & 2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & -1 & 2 \\
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
\vdots \\
y_n \\
\end{pmatrix}
= \begin{pmatrix}
g_{\beta,1} \\
g_{\beta,2} \\
g_{\beta,3} \\
\vdots \\
g_{\beta,n} \\
\end{pmatrix}.
\tag{1}
\]

We use the method of conjugate gradients to solve this problem. As the method of conjugate gradients performs identically quickly on all similar systems [3], rather than dealing with (1) directly, we deal with the equivalent problem (found using a similarity transform):

\[
B = \begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
\vdots \\
y_n \\
\end{pmatrix}
= \begin{pmatrix}
\begin{array}{cccc}
4 \sin^2 \left( \frac{j \pi}{2(n+1)} \right) \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & \ddots & \ddots \\
& & & & & \ddots & \ddots \\
& & & & & & \ddots & \ddots \\
& & & & & & & \ddots & \ddots \\
& & & & & & & & \ddots & \ddots \\
\end{array}
\end{pmatrix}
\begin{pmatrix}
f_\beta \quad B = \text{diag} \\
f_\beta \quad B = \text{diag} \\
f_\beta \quad B = \text{diag} \\
\vdots \\
f_\beta \quad B = \text{diag} \\
\end{pmatrix}.
\tag{2}
\]

We consider the solution vector \( \xi_{j,n} = \csc \left( \frac{j \pi}{2(n+1)} \right) \) (where the dependence of \( \xi_{j,n} \) on \( \beta \) has been suppressed). The larger \( \beta \) is, the smoother \( g_{\beta} \) is and the smaller \( \beta \) is the more jagged \( g_{\beta} \) is.

In the present paper, we start with a general analysis of the conjugate gradient method for uniformly bounded solutions vectors and for matrices whose eigenvalues are uniformly bounded and positive. We show that in such cases a fixed finite number of iterations gives some fixed amount of improvement as \( n \to \infty \). Then we specialize to (2). We show that in this case if \( \beta < 1.5 \) or \( \beta > 3.5 \), we see this effect. For the other values of \( \beta \) we find that the gain made by performing a fixed number of iterations of the method tends to zero as the size of the matrix tends towards infinity.

2. Uniformly bounded \( \xi_{j,n} \)

When dealing with the conjugate gradient method, it is standard to make all measurements in the \( B \)-norm—the energy norm. Throughout this paper, we follow this practice. Suppose, as is the case here, that the matrix \( B \) is symmetric and positive-definite. We define the \( B \)-norm as:

\[
\|v\|_B = (v, Bv).
\]

It is well known [3, p. 31] that when one uses the method of conjugate gradients, the square of the error (in the energy norm) of the solution after