New formulations of a Stokes type problem related to the primitive equations of the atmosphere and applications

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Received September 10, 1998 / Published online August 2, 2000 – © Springer-Verlag 2000

Summary. The aim of this article is to propose new algorithms for a Stokes type system related to the primitive equations of atmosphere, which are the fundamental equations for the motion of the atmosphere [6]. We derive an equivalent formulation of these equations in which the natural constraint appearing in these equations is automatically satisfied without being explicitly imposed. Numerical algorithms based on the new formulation appeared to be very competitive compared to the Uzawa-Conjugate Gradient method.

Mathematics Subject Classification (1991): 76M30

1. Introduction

In this article we recall a Stokes type problem related to the primitive equations of atmosphere [6], and we propose a new formulation of this problem. The system considered is the following:

\[
\begin{cases}
\lambda u - \nu \Delta_3 u + \text{grad} \Phi = f \text{ in } M, \\
\int_0^1 \text{div} u \, d\zeta = 0, \\
u = 0 \text{ on } \partial M,
\end{cases}
\]

(1.1)

where \( M = S^2 \times (0, 1) \), and \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \), \( \Delta_3 \) is the three-dimensional Laplace operator for vectors defined in [6] and recalled hereafter, \( u = u_\theta e_\theta + u_\varphi e_\varphi \) is the two-dimensional velocity, \( \Phi \) is the geopotential.

\[1\] We recall that, in the non Cartesian case, the Laplacian of a vector is not uniquely defined. Indeed, in differential geometry, several related definition appear (see e.g. [1,3,5]); each definition possesses part only of the properties of the Cartesian Laplacian.
The unknown function is \( u = u(\theta, \varphi, \zeta) \), where \( r, \theta, \varphi \) are the spherical coordinate in \( \mathbb{R}^3 \), \( \zeta \) is the pressure variable, and the geopotential \( \Phi \) is equal to \( gz \), where \( z = r - 1 \) (1 = radius of the sphere \( S^2 \)), and \( g \) is the gravitational constant. The function \( \Phi = \Phi(\theta, \varphi) \) plays also the role of the Lagrange multiplier for the constraint (second equation) in (1.1), [6]. The system (1.1) is related to the primitive equations of the atmosphere, which are the fundamental equations of the atmosphere, [6]; for \( \lambda = 0 \), (1.1) is exactly the linear dissipative operator appearing in these equations.

For the numerical resolution of (1.1), if we use classical methods such as the Uzawa-conjugate gradient method, we are confronted with the nonlocal constraint

\[
\int_0^1 \text{div} u \, d\zeta = 0,
\]

which is not easy to handle numerically.

In the new formulation proposed in this article, the constraint (1.2) is automatically satisfied without being explicitly imposed. Hence this procedure is well suited for numerical computations. Moreover, in the new formulation, the geopotential \( \Phi \) is not a variable anymore and it can be explicitly computed at the end, after the velocity has been found.

The main idea behind the new formulation is to split the velocity into the form

\[
u = v + \omega,
\]

where the vector functions

\[
v = v(\theta, \varphi) \quad \text{and} \quad \omega = \omega(\theta, \varphi, \zeta)
\]

satisfy a simple coupled system of equations.

In a different context, the decomposition (1.3) was also used in [9] in order to study the regularity of the solutions of (1.1), in the case of the primitive equations of the ocean (the geometrical domain is then different).

The article is organized as follows. In the next section, we recall some function spaces following [6, 7], and we propose a mixed variational formulation for (1.1). In the third section, we propose the new algorithms for (1.1), in which the constraint (1.2) is automatically satisfied without being explicitly imposed. The fourth section compares numerical results obtained with the new algorithms and with the Uzawa-Conjugate Gradient method.