Summary. In this paper we establish a $C^1$ error estimation on the boundary for the solution of an exterior Neumann problem in $\mathbb{R}^3$. To solve this problem we consider an integral representation which depends from the solution of a boundary integral equation. We use a full piecewise linear discretisation which on one hand leads to a simple numerical algorithm but on the other hand the error analysis becomes more difficult due to the singularity of the integral kernel. We construct a particular approximation for the solution of the boundary integral equation, for the solution of the Neumann problem and its gradient on the boundary and estimate their $C^0$ error.

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Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^2$ and $\mathring{\Omega} = \overline{\Omega}$ its exterior. Let also $\Gamma = \partial \Omega$ be the boundary of $\Omega$ and $\nu$ the unitary normal vector to $\Gamma$ and exterior to $\Omega$. We are interested in the approximation on the boundary of $u$ and $Du$ where $u$ is given by:

\begin{equation}
\begin{cases}
-\Delta u = 0 & \text{in } \mathring{\Omega}, \\
\frac{\partial u}{\partial \nu} = g & \text{on } \Gamma, \\
u = o(1) \text{ at infinity},
\end{cases}
\end{equation}

with $g \in C^\epsilon(\Gamma)$, $\epsilon \in [0, 1)$. 

$C^1$ error estimation on the boundary for an exterior Neumann problem in $\mathbb{R}^3$

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It is well-known that the problem (1) has a unique solution. Moreover, $u$ is given by an integral representation, see [2,5]:

$$u(x) = \int_{\Gamma} q(y)\Phi(x, y)dy, \ x \in \Omega,$$

where $\Phi(x, y) = \frac{1}{4\pi \lvert x - y \rvert}$ is the fundamental solution of Laplace’s equation in $\mathbb{R}^3$. The unknown function $q$ is the unique solution in $C^\epsilon(\Gamma)$ (see [2]) of the following integral equation on $\Gamma$:

$$q(x) - 2\int_{\Gamma} q(y)\frac{\partial\Phi(x, y)}{\partial\nu(x)}dy = -2g(x), \ x \in \Gamma,$$

where $\frac{\partial\Phi(x, y)}{\partial\nu(x)} = D^x\Phi(x, y) \cdot \nu(x)$, $D^x$ is the gradient operator with respect to $x$ and $\nu(x)$ is the normal vector $\nu$ at $x$. If $g \in C^\epsilon(\Gamma)$ with $\epsilon \in (0, 1)$ then $Du \in C^\epsilon(\Gamma; \mathbb{R}^3)$ and (see [2,5]):

$$Du(x) = -\frac{1}{2}q(x)\nu(x) + \int_{\Gamma} q(y)D^x\Phi(x, y)dy, \ x \in \Gamma,$$

where the above integral is understood in the sense of Cauchy’s principal value. Finally, if $I$ is the identity operator on $C^\epsilon(\Gamma)$ and $K : C^\epsilon(\Gamma) \mapsto C^\epsilon(\Gamma)$ is given by

$$Kq(x) = 2\int_{\Gamma} q(y)\frac{\partial\Phi(x, y)}{\partial\nu(x)}dy, \ x \in \Gamma,$$

then the equation (3) is equivalent to:

$$Aq = -2g, \quad A := I - K.$$

Our aim in this paper is to estimate the $C^0$ error approximation of $u$ and $Du$ on the boundary by using a full piecewise linear discretisation which means that we discretise the boundary $\Gamma$ and the functional space $C^0(\Gamma)$. This leads to a simple algorithm for the computation of approximation of $q$, $u$ and $Du$. On the other hand, the error analysis becomes more delicate because the integral kernel in the integrals above are singular and (3), (4) has no sense for a piecewise linear boundary.

The approximation of the exterior Neumann boundary problem using singular integral representation is object of many works, see for example [1,4,7,9,10]. In these works to obtain a $C^1(\Gamma)$ error estimate a high order approximation is required and, generally, the smoothness of the underlying boundary surface considered is higher than in our paper.