A boundary multiplier/fictitious domain method
for the steady incompressible Navier-Stokes equations

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Received February 24, 1999 / Revised version received January 30, 2000 /
Published online October 16, 2000 – © Springer-Verlag 2000

\textbf{Summary.} We analyze the error of a fictitious-domain method with boundary Lagrange multiplier. It is applied to solve a non-homogeneous steady incompressible Navier-Stokes problem in a domain with a multiply-connected boundary. The interior mesh in the fictitious domain and the boundary mesh are independent, up to a mesh-length ratio.

\textit{Mathematics Subject Classification (1991):} 65D30, 65N15, 65N30.

\textbf{0. Introduction}

There is presently a marked interest for fictitious-domain methods as a tool for solving complex boundary-value problems from Science and Engineering; see for instance [12] and the references therein for examples and applications. The main reason for this growing popularity is that these methods allow the use of fairly structured meshes on a simply-shaped auxiliary domain containing the actual one, thus permitting the use of fast solvers.

The present article can be viewed as a theoretical justification of the methods described and implemented in [13] for the numerical simulation

\textsuperscript{⋆} Part of this work was performed while the author was visiting the Department of Mathematics of the University of Houston
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of incompressible viscous flow modelled by the Navier-Stokes equations. Here, we shall discuss the solution of the steady incompressible Navier-Stokes equations with non-homogeneous Dirichlet boundary conditions, by a methodology combining finite-element approximations with boundary multipliers on a fictitious domain.

More precisely, let \( \omega \) be a bounded domain of \( \mathbb{R}^2 \) with a Lipschitz-continuous boundary \( \gamma \), that is not necessarily connected, with connected components \( \gamma_i \), \( 0 \leq i \leq m \). The steady-state, non-homogeneous, incompressible Navier-Stokes problem is: find a velocity vector \( \mathbf{u} \) and a scalar pressure \( p \) such that

\[
-\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \omega, \\
\text{div} \mathbf{u} = 0 \quad \text{in} \quad \omega, \\
\mathbf{u} = \mathbf{g} \quad \text{on} \quad \gamma.
\]

The assumptions on the data are: \( \nu > 0 \) constant, \( \mathbf{f} \in L^2(\omega)^2 \) and \( \mathbf{g} \in H^{1/2}(\gamma)^2 \) with

\[
\int_{\gamma_i} \mathbf{g} \cdot \mathbf{n} \, d\sigma = 0, \quad 0 \leq i \leq m,
\]

where \( \mathbf{n} = (n_1, n_2) \) denotes the unit normal vector to each \( \gamma_i \), pointing outside \( \omega \). Using Leray-Hopf’s Lemma (cf. for instance Hopf [15], Lions [17] or Girault & Raviart [11]), this last assumption guarantees that (0.1)–(0.3) has at least one solution \( \mathbf{u} \in H^1(\omega)^2 \) and \( p \in L^2(\omega) \), without restriction on the size of the data. In fact, (0.1)–(0.3) has at least one solution for a wider range of data, but we assume smoother data in view of the discretization.

The fictitious-domain method consists in imbedding \( \omega \) in a larger domain \( \Omega \) that has a very simple shape, solving numerically the Navier-Stokes equations in \( \Omega \) by a finite-element method and recovering the boundary condition on \( \gamma \) by a Lagrange multiplier. Here, we find that this multiplier is not unique unless it satisfies some orthogonality condition with the normal vector to each connected component \( \gamma_i \) of the boundary. In the conjugate-gradient algorithm described in [13], this condition is imposed by starting with an adequate initial value for the Lagrange multiplier. As in previous works using this fictitious-domain method, the interior mesh in \( \Omega \) and the boundary mesh on \( \gamma \) are unrelated up to a mesh-length ratio (cf. Girault & Glowinski [10]). In this article, we shall prove convergence and error estimates for this method under adequate sufficient conditions on the solution and the mesh-length ratio. The inf-sup condition corresponding to the divergence and boundary value constraints is established as in Babuška [3] and Fallah [9], and the non-linearity is handled by the implicit function argument of Brezzi, Rappaz & Raviart [6].