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Renormalization in Quantum Field Theory and the Riemann–Hilbert Problem II: The $\boldsymbol{\beta}$-Function, Diffeomorphisms and the Renormalization Group*

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Abstract: We showed in Part I that the Hopf algebra $\mathcal{H}$ of Feynman graphs in a given QFT is the algebra of coordinates on a complex infinite dimensional Lie group G and that the renormalized theory is obtained from the unrenormalized one by evaluating at $\varepsilon = 0$ the holomorphic part $\gamma_+(\varepsilon)$ of the Riemann–Hilbert decomposition $\gamma_-(\varepsilon)^{-1}\gamma_+(\varepsilon)$ of the loop $\gamma(\varepsilon) \in G$ provided by dimensional regularization. We show in this paper that the group G acts naturally on the complex space X of dimensionless coupling constants of the theory. More precisely, the formula $g_0 = gZ_1Z_3^{-3/2}$ for the effective coupling constant, when viewed as a formal power series, does define a Hopf algebra homomorphism between the Hopf algebra of coordinates on the group of formal diffeomorphisms to the Hopf algebra $\mathcal{H}$. This allows first of all to read off directly, without using the group G, the bare coupling constant and the renormalized one from the Riemann–Hilbert decomposition of the unrenormalized effective coupling constant viewed as a loop of formal diffeomorphisms. This shows that renormalization is intimately related with the theory of non-linear complex bundles on the Riemann sphere of the dimensional regularization parameter ε. It also allows to lift both the renormalization group and the β-function as the asymptotic scaling in the group G. This exploits the full power of the Riemann–Hilbert decomposition together with the invariance of $\gamma_-(\varepsilon)$ under a change of unit of mass. This not only gives a conceptual proof of the existence of the renormalization group but also delivers a scattering formula in the group G for the full higher pole structure of minimal subtracted counterterms in terms of the residue.

1. Introduction

We showed in Part I of this paper [1] that perturbative renormalization is a special case of a general mathematical procedure of extraction of finite values based on the Riemann–Hilbert problem. More specifically we associated to any given renormalizable quantum

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field theory an (infinite dimensional) complex Lie group G. We then showed that passing from the unrenormalized theory to the renormalized one was exactly the replacement of the loop $d \to \gamma(d) \in G$ of elements of G obtained from dimensional regularization (for $d \neq D =$ dimension of space-time) by the value $\gamma_+(D)$ of its Birkhoff decomposition, $\gamma(d) = \gamma_-(d)^{-1}\,\gamma_+(d)$.

The original loop $d \to \gamma(d)$ not only depends upon the parameters of the theory but also on the additional "unit of mass" μ required by dimensional analysis. We shall show in this paper that the mathematical concepts developed in Part I provide very powerful tools to lift the usual concepts of the β-function and renormalization group from the space of coupling constants of the theory to the complex Lie group G.

We first observe, taking φ_6^3 as an illustrative example to fix ideas and notations, that even though the loop $\gamma(d)$ does depend on the additional parameter μ,

$$\mu \to \gamma_\mu(d), \tag{1}$$

the negative part γ_{μ^-} in the Birkhoff decomposition,

$$\gamma_\mu(d) = \gamma_{\mu^-}(d)^{-1}\,\gamma_{\mu^+}(d) \tag{2}$$

is actually independent of μ,

$$\frac{\partial}{\partial\mu}\,\gamma_{\mu^-}(d) = 0. \tag{3}$$

This is a restatement of a well known fact and follows immediately from dimensional analysis. Moreover, by construction, the Lie group G turns out to be graded, with grading,

$$\theta_t \in \operatorname{Aut} G\ , \quad t \in \mathbb{R}, \tag{4}$$

inherited from the grading of the Hopf algebra $\mathcal{H}$ of Feynman graphs given by the loop number,

$$L(\Gamma) = \text{loop number of } \Gamma \tag{5}$$

for any 1PI graph Γ.

The straightforward equality,

$$\gamma_{e^t\mu}(d) = \theta_{t\varepsilon}(\gamma_\mu(d)) \qquad \forall\, t \in \mathbb{R},\ \varepsilon = D - d \tag{6}$$

shows that the loops γ_μ associated to the unrenormalized theory satisfy the striking property that the negative part of their Birkhoff decomposition is unaltered by the operation,

$$\gamma(\varepsilon) \to \theta_{t\varepsilon}(\gamma(\varepsilon)). \tag{7}$$

In other words, if we replace $\gamma(\varepsilon)$ by $\theta_{t\varepsilon}(\gamma(\varepsilon))$ we do not change the negative part of its Birkhoff decomposition. We settle now for the variable,

$$\varepsilon = D - d \in \mathbb{C}\backslash\{0\}. \tag{8}$$

Our first result (Sect. 2) is a complete characterization of the loops $\gamma(\varepsilon) \in G$ fulfilling the above striking invariance. This characterization only involves the negative part $\gamma_-(\varepsilon)$ of their Birkhoff decomposition which by hypothesis fulfills,

$$\gamma_-(\varepsilon)\,\theta_{t\varepsilon}(\gamma_-(\varepsilon)^{-1}) \text{ is convergent for } \varepsilon \to 0. \tag{9}$$