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Minimal 2-spheres in $\mathbb{C}P^n$ with constant Kähler angle

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Abstract. In 1988 the author and J. Bolton conjectured that a minimally immersed 2-sphere in $\mathbb{C}P^n$ with constant Kähler angle $\theta \neq 0, \pi/2, \pi$ necessarily has constant curvature. In 1995 Li Zhen-qi showed that the simplest candidates for counterexamples must be linearly full in $\mathbb{C}P^{10}$ with $\tan^2(\theta/2) = 3/4$, and produced an explicit 3-parameter family of them. In the present paper it is shown that these counterexamples may be completely characterised using almost complex curves in the nearly Kähler $S^6$ and that the space of such counterexamples, modulo ambient isometries, is a 14-cell with a single point removed.

1. Introduction

In [3] the following conjecture was made.

**Conjecture.** Let $\psi : S^2 \to \mathbb{C}P^n$ be a linearly full conformal minimal immersion with constant Kähler angle, and suppose that $\psi$ is neither holomorphic, anti-holomorphic nor totally real. Then, up to holomorphic isometries of $\mathbb{C}P^n$, $\psi$ belongs to the Veronese sequence.

Here and throughout we assume that $\mathbb{C}P^n$ is equipped with the Fubini-Study metric of constant holomorphic curvature 4. We recall that the Veronese sequence is the Frenet frame $\psi_0, \ldots, \psi_n$ of the holomorphic curve $\psi_0 : S^2 \to \mathbb{C}P^n$ defined by $\psi_0(z) = [1, \ldots, \sqrt{\binom{n}{k}} z^k, \ldots, z^n]$ and each such $\psi_p$ has both constant curvature and constant Kähler angle. Moreover, up to holomorphic isometries of $\mathbb{C}P^n$, any linearly full harmonic map $\psi : S^2 \to \mathbb{C}P^n$ with induced metric of constant Gaussian curvature is an element of the Veronese sequence [1].

It was shown in [3] that the conjecture is true if $\psi$ is totally unramified and $n$ and $n + 2$ are consecutive prime numbers. However, in [8] Li Zhen-qi showed that the conjecture as stated is false. He showed that if $\psi_0, \ldots, \psi_n$ is the Frenet frame of a totally unramified linearly full holomorphic curve $\psi_0 : S^2 \to \mathbb{C}P^n$ and $\psi_2$
has constant Kähler angle $\theta_2$ but non-constant Gaussian curvature then $n = 10$ and $\tan^2 \frac{\theta_2}{2} = \frac{3}{4}$. Furthermore he produced a 3-parameter family of such maps.

The purpose of this article is to show how these results may be considerably strengthened. Somewhat surprisingly the characterisation of such counterexamples may be given in terms of almost complex curves in the nearly Kähler $S^6$ and the exceptional group $G_2$. In particular, we prove the following.

**Theorem.** Let $\psi_0 : S^2 \to \mathbb{C}P^n$ be a linearly full holomorphic curve with Frenet frame $\psi_0, \ldots, \psi_n$ and suppose that $\psi_2$ has constant Kähler angle $\theta_2$ and that the ramification indices $r(\partial_0), r(\partial_1)$ are both zero. Then $\psi_0$ is either

(i) of constant curvature; or

(ii) the first osculating curve of the directrix curve of a totally unramified linearly full almost complex curve in the nearly Kähler $S^6$.

In case (ii) $\psi_0, \ldots, \psi_n$ is totally unramified, $n = 10$ and $\tan^2 \frac{\theta_2}{2} = \frac{3}{4}$, and the space of such $\psi_0$ modulo holomorphic isometries of $\mathbb{C}P^{10}$ is the 14-cell $G_{14}/G_2$. Furthermore, with the exception of one point, every element of this cell corresponds to a map of non-constant curvature.

### 2. Preliminaries

In this section we gather together some basic results on holomorphic curves in $\mathbb{C}P^n$ further details of which can be found in [3] and [7]. Throughout $S$ will denote a Riemann surface.

Let $\psi_0 : S \to \mathbb{C}P^n$ be a linearly full holomorphic curve and let $\psi_0, \ldots, \psi_n$ be the corresponding Frenet frame. Thus if $\psi_0$ is expressed in terms of a local complex coordinate $z$ as $\psi_0(z) = [f_0(z)]$, where $f_0 : U \to \mathbb{C}^{n+1} \setminus \{0\}$ is holomorphic and $U$ is a simply connected open subset of $S$, then

$$\psi_p(z) = [f_p(z)], \quad p = 0, \ldots, n,$$

for mutually orthogonal $\mathbb{C}^{n+1}$-valued functions $f_0, \ldots, f_n$ satisfying

$$\frac{\partial f_p}{\partial z} = f_{p+1} + \frac{\partial}{\partial z} \log |f_p|^2 f_p, \quad (2.1)$$

$$\frac{\partial f_p}{\partial \bar{z}} = -\frac{|f_p|^2}{|f_{p-1}|^2} f_{p-1}, \quad (2.2)$$

where we set $f_{n+1} = 0$ when $p = n$ in (2.1) and the right hand side of (2.2) is zero for $p = 0$. From these equations it follows that

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log |f_p|^2 = \frac{|f_{p+1}|^2}{|f_p|^2} - \frac{|f_{p-1}|^2}{|f_p|^2} \quad (2.3)$$

and hence

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log |f_0 \wedge \ldots \wedge f_p|^2 = \frac{|f_{p+1}|^2}{|f_p|^2} \quad (2.4).$$