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$L^p$ estimates for the Cauchy–Riemann operator on $q$-convex intersections in $\mathbb{C}^n$

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Abstract. We construct a new solution operator for $\bar{\partial}$ on certain piecewise smooth $q$-convex intersections. $L^p$ estimates are obtained for the solution operators of $\bar{\partial}$-closed forms on such domains.

Introduction

Great progress has been made recently in understanding the $\bar{\partial}$-Neumann problem on piecewise smooth domains. Henkin–Iordan–Kohn [8], Michel-Shaw [15] obtained subelliptic $\frac{1}{2}$-estimates for the $\bar{\partial}$-Neumann operator on piecewise smooth intersections of strongly pseudoconvex domains. Henkin–Iordan, [7] showed compactness of the $\bar{\partial}$-Neumann operator on bounded pseudoconvex domains $D$ with B-regular boundary (i.e. there exists a continuous function $\rho$ in a neighborhood $U$ of $\partial D$ such that $D \cap U = \{ \rho < 0 \}$, $dd^c \rho \geq dd^c |z|^2$ where $d^c = \frac{i}{4\pi} (\partial - \bar{\partial})$).

Straube [17] obtained subelliptic $\delta$ estimates ($\delta < \frac{1}{2}$) for piecewise smooth intersections of finite 1-D’Angelo type domains. The key ingredient in the proof of all of the results above is an exhaustion of the piecewise smooth domain by smooth (or uniformly Lipschitz) strongly pseudoconvex domains on which the $\bar{\partial}$-Neumann operators exist and satisfy uniform $L^2$ or subelliptic $\epsilon$-estimates.

Much less is known when the domains are not pseudoconvex. Due to its intimate connection to the tangential Cauchy–Riemann operator operator it would be of great interest to understand the $\bar{\partial}$-Neumann operator on more general piecewise smooth domains that arise as piecewise smooth intersections of smooth $q$-convex domains.

Definition 1. A bounded smooth domain $D$ in $\mathbb{C}^n$ is called strongly $q$-convex (resp. weakly $q$-convex) if there exists a bounded neighborhood $W$ of $\partial D$ and a smooth defining function $r$ of $D$ such that for every $z \in \partial D$ the Levi form of $r|_{\partial D}$ at $z$ has at least $n - q$ positive (resp. non-negative) eigenvalues.
In particular, a strongly 1-convex domain is strongly pseudoconvex.

The category of \( q \)-convex domains when \( q > 1 \) is still rather unexplored. In contrast to the pseudoconvex case, there are major differences between 1-convex and \( q \)-convex domains when \( q > 1 \).

\( \alpha \) The intersection of any two \( q \)-convex domains (\( q > 1 \)) is not necessarily \( q \)-convex (that will lead to problems in smoothing \( q \)-subharmonic functions).

\( \beta \) There are no \( L^2 \) estimates for \( \partial \) with “good” constants on smooth strongly \( q \)-convex domains and there are no unweighted \( L^2 \) estimates for \( \partial \) on smooth weakly \( q \)-convex domains.

\( \gamma \) There are no global holomorphic support functions for smooth strongly \( q \)-convex domains that can be used to construct global integral solution operators for \( \partial \) on such domains.

In this paper we shall study the \( \partial \) and \( \bar{\partial} \)-Neumann problem for the following type of piecewise smooth domains:

**Definition 2.** A bounded domain \( \Omega \subset \subset \mathbb{C}^n \) shall be called a \( C^3 \) \( q \)-convex intersection if there exists a bounded neighborhood \( W \) in \( \mathbb{C}^n \) of \( \Omega \) and a finite number of real \( C^3 \) functions \( \rho_1, \ldots, \rho_N \) where \( n \geq N + 2 \) defined on \( W \) such that \( \Omega = \{ z \in W | \rho_1(z) < 0, \ldots, \rho_N(z) < 0 \} \) and the following are true:

\( i \) For \( 1 \leq i_1 < i_2 < \cdots < i_\ell \leq N \) the 1-forms \( d\rho_i, \ldots, d\rho_\ell \) are \( \mathbb{R} \)-linearly independent on \( \bigcap_{j=1}^{\ell} \{ \rho_j \leq 0 \} \).

\( ii \) For \( 1 \leq i_1 < \cdots < i_\ell \leq N \), for every \( z \in \bigcap_{j=1}^{\ell} \{ \rho_j \leq 0 \} \), if we set \( I = (i_1, \ldots, i_\ell) \), there exists a linear subspace \( T^I_\mathbb{C} \) of \( \mathbb{C}^n \) of complex dimension at least \( n - q + 1 \) such that for \( i \in I \) the Levi forms \( L\rho_i \) restricted on \( T^I_\mathbb{C} \) are positive definite.

Condition ii) was introduced by Grauert [5]. It implies that at every “corner” the Levi forms of the corresponding \( \{ \rho_i \} \) have their positive eigenvalues along the same directions.

Thiebaut–Leiterer [13] solved the \( \bar{\partial} \)-problem with Hölder estimates for piecewise smooth intersections of \( q \)-convex domains where instead of condition ii) they required that the Levi form of any nontrivial convex combination of \( \{ \rho_i \}_{i=1}^N \) has at least \( n - q + 1 \) positive eigenvalues. These type of domains were originally considered by Henkin [6]. However, their solution operators are not suitable for proving \( L^3 \) (or more generally \( L^p, 1 \leq p \leq \infty \)) estimates. In this paper we construct a different solution operator for \( \bar{\partial} \) by means of Berndtsson–Andersson operators with multiple weights. The idea of multiple weights appeared in a paper of Berndtsson [3] on a division and interpolation problem in \( \mathbb{C}^n \).

The main results are the following theorems:

**Theorem 1.** Let \( \Omega \subset \subset \mathbb{C}^n \) be a \( C^3 \) \( q \)-convex intersection. Let \( p, q, s \in \mathbb{N}, 1 \leq q \leq n, 1 \leq p \leq \infty, s \geq q \). Given \( f \in L^p_{0,s}(\Omega) \bar{\partial}f = 0 \) in \( \Omega \), there exists \( u \in L^p_{0,s-1}(\Omega) \) such that \( \bar{\partial}u = f \) in \( \Omega \). More precisely we have:

\[
\|u\|_{L^p(\Omega)} \leq c_1 \|f\|_{L^p(\Omega)}
\]

where the constant \( c_1 \) is independent of \( f \).