RESEARCH ARTICLE

On Equalizer-flat Acts

Wolfram Bentz and Sydney Bulman-Fleming*

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Abstract

In 1971, B. Stenström characterized what are now called strongly flat acts over a monoid $S$: a right $S$-act $A$ is strongly flat if and only if the functor $A \otimes -$ (from the category of left $S$-acts into the category of sets) preserves both pullbacks and equalizers. Stenström gave two interpolation-type conditions whose conjunction describes strong flatness. In 1986, P. Normak studied these conditions separately, labelling them $(P)$ and $(E)$, and investigated their relation to different types of flatness. Bulman-Fleming, in 1991, showed that pullback flatness and strong flatness actually coincide, and several papers have appeared in which condition $(P)$ is discussed. To date, little work has been done on equalizer flat acts. This paper gives some new results connecting condition $(E)$ and equalizer flatness, concentrating on situations in which the two coincide. A description is given of the completely simple and completely 0-simple semigroups (with 1 adjoined) over which this occurs.

1. Introduction and preliminaries

Let $S$ be a monoid. The definitions of left and right $S$-acts, tensor products of acts, and basic results on flatness of acts can be found for example in [1] and will not be repeated here. We will however mention a few notions that are less well known. (For basic concepts in semigroup theory, in particular completely ($0-$) simple semigroups and their Rees matrix representations, the reader should consult [4], for example.) A right $S$-act $A$ (often denoted $A_S$ in the sequel) is called equalizer flat if the functor $A_S \otimes -$ (from the category of left $S$-acts to the category of sets) preserves equalizers, and flat if this functor preserves all monomorphisms. A monoid $S$ is called (left, right) absolutely flat if all (left, right) $S$-acts are flat. Without loss of generality we will assume that the equalizer of $f,g : B \longrightarrow C$ (interpreted in either the category of left $S$-acts or in the category of sets) is $E = \{ b \in B : f(b) = g(b) \}$ (where for convenience we allow the empty left $S$-act if necessary). Thus, $A_S$ is equalizer flat if and only if, for any equalizer $E$ of left acts as described above, the mapping $\psi : A \otimes E \longrightarrow E$ is a bijection, where $E = \{ a \otimes b \in A_S \otimes S B : a \otimes f(b) = a \otimes g(b) \text{ in } A_S \otimes S C \}$, and $\psi(a \otimes e) = a \otimes e$ for every $a \in A, e \in E$.

In [6], a right $S$-act is defined to satisfy condition $(E)$ if, whenever $a \in A$ and $s.t \in S$ are such that $as = at$, there exist $\bar{a} \in A$ and $u \in S$ such that $a = \bar{a}u$ and $us = ut$. It is shown in [6] that equalizer flat implies condition $(E)$, but not conversely, and that equalizer flat implies flat, but not conversely. Liu [5] has shown that every right $S$-act satisfying condition $(E)$ is flat if and only if $S$ is a regular monoid. The main purpose of the present paper is to further investigate connections among these three properties, paying special attention to monoids over which all right acts satisfying condition $(E)$ are in fact equalizer flat.

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We first present a criterion for a right act to be equalizer flat that will be used extensively throughout this paper. (We remark that an interpolation-type condition for this, in the spirit of conditions (E) and (P) of [6], or the condition (PF) presented in [3], is not yet known.)

**Proposition 1.1.** Let $S$ be a monoid and let $A$ be a right $S$-act. Then $A$ is equalizer-flat if and only if

(i) $A$ is flat, and

(ii) for every left $S$-act $C$, and for every $a \in A$, $c, c' \in C$, $a \otimes c = a \otimes c'$ in $A \otimes C$ implies $a = \tilde{a}u$ and $uc = uc'$ for some $\tilde{a} \in A, u \in S$.

**Proof.** Let $A$ be equalizer flat. Then, as noted earlier, equalizer flat implies flat, so (i) is clear. To establish (ii), let $C$ be any left $S$-act, and let $a \in A, c, c' \in C$ be elements for which $a \otimes c = a \otimes c'$ in $A \otimes C$. Consider the equalizer diagram

$$
\begin{array}{ccc}
S_E & \xrightarrow{\iota} & S_S \\
\downarrow f & & \downarrow g \\
S_C & & 
\end{array}
$$

in the category of left $S$-acts, where $f(s) = sc$ and $g(s) = sc'$ for each $s \in S$. (We follow the convention announced in the previous section concerning equalizers, and assume $\iota$ is inclusion.) As $A$ is equalizer flat, we have the equalizer diagram

$$
A \otimes E \xrightarrow{id_A \otimes \iota} A_S \otimes S_S \\
\downarrow id_A \otimes f & & \downarrow id_A \otimes g \\
A_S \otimes S_C
$$

in the category of sets. Since $(id_A \otimes f)(a \otimes 1) = a \otimes c = a \otimes c' = (id_A \otimes g)(a \otimes 1)$, it follows that $a \otimes 1$ belongs to $A \otimes E$, and so $a \otimes 1 = \tilde{a} \otimes u$ in $A \otimes S$ for some $\tilde{a} \in A$ and $u \in E$. Thus, $uc = uc'$, and using the standard isomorphism between $A \otimes S$ and $A$, it follows that $a = \tilde{a}u$.

Now assume $A$ satisfies conditions (i) and (ii), and let

$$
\begin{array}{ccc}
S_E & \xrightarrow{\iota} & S_M \\
\downarrow f & & \downarrow g \\
S_N & & 
\end{array}
$$

be an equalizer diagram in the category of left $S$-acts. Letting $E$ be the equalizer in the diagram

$$
E \xrightarrow{\iota'} A_S \otimes S_M \xrightarrow{id_A \otimes f} A_S \otimes S_N,
$$

we must show that the mapping $\psi : A \otimes E \longrightarrow E$ defined by $\psi(a \otimes e) = a \otimes e$ ($a \in A, e \in E$) is a bijection. In the commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\iota'} & A_S \otimes S_M \\
\downarrow \psi & & \downarrow id_A \otimes \iota \\
A_S \otimes S_E & & A_S \otimes S_E
\end{array}
$$

we have $\psi$ is a bijection.