RESEARCH ARTICLE

On Equalizer-flat Acts

Wolfram Bentz and Sydney Bulman-Fleming*

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Abstract

In 1971, B. Stenström characterized what are now called strongly flat acts over a monoid $S$: a right $S$-act $A$ is strongly flat if and only if the functor $A \otimes -$ (from the category of left $S$-acts into the category of sets) preserves both pullbacks and equalizers. Stenström gave two interpolation-type conditions whose conjunction describes strong flatness. In 1986, P. Normak studied these conditions separately, labelling them (P) and (E), and investigated their relation to different types of flatness. Bulman-Fleming, in 1991, showed that pullback flatness and strong flatness actually coincide, and several papers have appeared in which condition (P) is discussed. To date, little work has been done on equalizer flat acts. This paper gives some new results connecting condition (E) and equalizer flatness, concentrating on situations in which the two coincide. A description is given of the completely simple and completely 0-simple semigroups (with 1 adjoined) over which this occurs.

1. Introduction and preliminaries

Let $S$ be a monoid. The definitions of left and right $S$-acts, tensor products of acts, and basic results on flatness of acts can be found for example in [1] and will not be repeated here. We will however mention a few notions that are less well known. (For basic concepts in semigroup theory, in particular completely (0-) simple semigroups and their Rees matrix representations, the reader should consult [4], for example.) A right $S$-act $A$ (often denoted $A_S$ in the sequel) is called equalizer flat if the functor $A_S \otimes -$ (from the category of left $S$-acts to the category of sets) preserves equalizers, and flat if this functor preserves all monomorphisms. A monoid $S$ is called (left, right) absolutely flat if all (left, right) $S$-acts are flat. Without loss of generality we will assume that the equalizer of $f,g:B \rightarrow C$ (interpreted in either the category of left $S$-acts or in the category of sets) is $E = \{b \in B \mid f(b) = g(b)\}$ (where for convenience we allow the empty left $S$-act if necessary). Thus, $A_S$ is equalizer flat if and only if, for any equalizer $E$ of left acts as described above, the mapping $\psi:A \otimes E \rightarrow E$ is a bijection, where $E = \{a \otimes b \in A_S \otimes S B : a \otimes f(b) = a \otimes g(b) \in A_S \otimes S C\}$, and $\psi(a \otimes e) = a \otimes e$ for every $a \in A, e \in E$.

In [6], a right $S$-act is defined to satisfy condition (E) if, whenever $a \in A$ and $s,t \in S$ are such that $as = at$, there exist $\bar{a} \in A$ and $u \in S$ such that $a = \bar{a}u$ and $us = ut$. It is shown in [6] that equalizer flat implies condition (E), but not conversely, and that equalizer flat implies flat, but not conversely. Liu [5] has shown that every right $S$-act satisfying condition (E) is flat if and only if $S$ is a regular monoid. The main purpose of the present paper is to further investigate connections among these three properties, paying special attention to monoids over which all right acts satisfying condition (E) are in fact equalizer flat.

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We first present a criterion for a right act to be equalizer flat that will be used extensively throughout this paper. (We remark that an interpolation-type condition for this, in the spirit of conditions (E) and (F) of [6], or the condition (PF) presented in [3], is not yet known.)

**Proposition 1.1.** Let $S$ be a monoid and let $A$ be a right $S$-act. Then $A$ is equalizer-flat if and only if

(i) $A$ is flat, and

(ii) for every left $S$-act $C$, and for every $a \in A$, $c, c' \in C$, $a \otimes c = a \otimes c'$ in $A \otimes C$ implies $a = \bar{a}u$ and $uc = uc'$ for some $\bar{a} \in A, u \in S$.

**Proof.** Let $A$ be equalizer flat. Then, as noted earlier, equalizer flat implies flat, so (i) is clear. To establish (ii), let $C$ be any left $S$-act, and let $a \in A, c, c' \in C$ be elements for which $a \otimes c = a \otimes c'$ in $A \otimes C$. Consider the equalizer diagram

$$
\begin{array}{ccc}
  sE & \xrightarrow{\iota} & sS \\
  f & \downarrow & g \\
  & sC 
\end{array}
$$

in the category of left $S$-acts, where $f(s) = sc$ and $g(s) = sc'$ for each $s \in S$. (We follow the convention announced in the previous section concerning equalizers, and assume $\iota$ is inclusion.) As $A$ is equalizer flat, we have the equalizer diagram

$$
\begin{array}{ccc}
  A \otimes E & \xrightarrow{id_A \otimes \iota} & A \otimes sS \\
  id_A \otimes f & \downarrow & id_A \otimes g \\
  & A \otimes sC 
\end{array}
$$

in the category of sets. Since $(id_A \otimes f)(a \otimes 1) = a \otimes c = a \otimes c' = (id_A \otimes g)(a \otimes 1)$, it follows that $a \otimes 1$ belongs to $A \otimes E$, and so $a \otimes 1 = \bar{a} \otimes u$ in $A \otimes S$ for some $\bar{a} \in A$ and $u \in E$. Thus, $uc = uc'$, and using the standard isomorphism between $A \otimes S$ and $A$, it follows that $a = \bar{a}u$.

Now assume $A$ satisfies conditions (i) and (ii), and let

$$
\begin{array}{ccc}
  sE & \xrightarrow{\iota} & sM \\
  f & \downarrow & g \\
  & sN 
\end{array}
$$

be an equalizer diagram in the category of left $S$-acts. Letting $E$ be the equalizer in the diagram

$$
\begin{array}{ccc}
  E & \xrightarrow{\iota'} & A \otimes sM \\
  id_A \otimes f & \downarrow & id_A \otimes g \\
  & A \otimes sN, 
\end{array}
$$

we must show that the mapping $\psi : A \otimes E \rightarrow E$ defined by $\psi(a \otimes e) = a \otimes e$ $(a \in A, e \in E)$ is a bijection. In the commutative diagram

$$
\begin{array}{ccc}
  E & \xrightarrow{\iota'} & A \otimes sM \\
  \psi & \downarrow & id_A \otimes \iota \\
  & A \otimes sE \\
  id_A \otimes f & \downarrow & id_A \otimes g \\
  & A \otimes sN, 
\end{array}
$$

we have