System Equivalence for AR-Systems over Rings—with an Application to Delay-Differential Systems*

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Abstract. In this paper the notion of autoregressive systems over an integral domain \( \mathcal{A} \) is introduced, as a generalization of AR-systems over the rings \( \mathbb{R}[s] \) and \( \mathbb{R}[s, s^{-1}] \). The interpretation of the dynamics represented by a matrix over \( \mathcal{A} \) is fixed by the choice of a module \( \mathcal{A} \) over \( \mathcal{A} \), consisting of all time-trajectories under consideration. In this setup the problem of system equivalence is studied: when do two different AR-representations characterize the same behavior? This problem is solved using a ring extension of \( \mathcal{A} \), that explicitly depends on the choice of the module \( \mathcal{A} \) of all time-trajectories. In this way the usual divisibility conditions on the system defining matrices can be recovered. The results apply to the class of delay-differential systems with (in)commensurable delays. In this particular application, the ring extension of \( \mathcal{A} \) is characterized explicitly.

Key words. Autoregressive (AR)-systems, Systems over rings, Delay-differential systems, Behavioral approach, System equivalence, Exponential polynomials.

1. Introduction

In the behavioral approach to dynamical systems, introduced by Willems (see, e.g., [W1] and [W2]), a system is described by a triple \( \langle T, W, \mathcal{B} \rangle \). Here \( T \) is the time-axis, \( W \) is the space in which the signals take their values, and \( \mathcal{B} \)—the behavior—is a subspace of the signal space \( W^T \). The behavior \( \mathcal{B} \) can be seen as the set of all time-trajectories, satisfying the laws governing the system.

As an example, consider a dynamical system, described by a set of ordinary differential equations (for continuous-time systems) or difference equations (for discrete-time systems), together with some nondynamic linear relations among the variables. In these situations, the behavior is the set of all time-trajectories satisfying the system defining equations. Collecting the variables in a vector \( w \), and the equations in a polynomial matrix \( P(s) \in \mathbb{R}[s]^{p \times d} \), the behavior is described by the set \( \{ w \mid P(d/dt)w = 0 \} \) for continuous-time systems, or \( \{ w \mid P(\sigma)w = 0 \} \) in the discrete-time case, where \( \sigma \) denotes the shift operator \( \sigma(x(t)) = x(t - 1) \). In both

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situations we see that the behavior is the kernel of an operator, described by a polynomial matrix in the differentiation operator $d/dt$ or the shift operator $σ$. A system described by such a kernel representation is called an autoregressive (AR)-system.

In this paper we study AR-systems over rings more general than $\mathbb{R}[s]$ or $\mathbb{R}[s,s^{-1}]$. In fact, we assume that the system defining equations are described by matrices over an arbitrary integral domain $Ω$ (cf. [H1]). In this way we are able to study classes of systems described by a more general type of dynamic equations. For example, delay-differential systems with (in)commensurable time-delays fit into this framework.

This generalization from the rings $\mathbb{R}[s]$ and $\mathbb{R}[s,s^{-1}]$ to arbitrary integral domains resembles in a way the extension of the theory of state space systems over fields to the ring case (see, e.g., [BBV] and [S3]). However, there is an important difference. In the theory of state-space systems over rings, the system defining equations are studied in a rather formal way, without fixing explicitly the context in which these equations should be interpreted. For example, the same quadruple of matrices, representing a discrete-time system over a polynomial ring in state-space form, can be used to describe a continuous delay system with point delays. In this approach, formal manipulations of the system defining equations are emphasized, and these transformations are applicable independent of the way the equations are interpreted.

For the problem studied in this paper—system equivalence—the context in which the defining equations should be interpreted is important. This context is fixed by defining a module $M$ over the ring $Ω$, describing the class of all (one-variable) time-trajectories under consideration. Each ring element corresponds to an operator acting on the elements of the module $M$. In the multivariable case, $M^Ω$ takes the place of the signal space $W^T$, and the dynamics of an AR-system over the ring $Ω$ is described by a matrix in $Ω^{p×q}$. Our goal is to find conditions on the ring $Ω$ and the module $M$, such that relevant system theoretic properties can be put in terms of conditions on the matrices over $Ω$, representing a system.

In comparison with the behavioral approach, the setup presented in this paper is slightly different. Instead of explicitly describing a time-axis $T$, and a space $W$ in which the signals take their values, we take $M^Ω$ as the signal space, with $M$ a module over the ring $Ω$. In this way it is possible to endow the signal space with a richer structure. Furthermore, the action of the operator corresponding to a ring element of $Ω$, is seen as an action on a time-trajectory in $M$ as a whole.

In this setting we study the problem of system equivalence: when are the behaviors described by two different AR-representations the same? For AR-systems over the polynomial ring $\mathbb{R}[s]$, representing continuous-time systems described by sets of linear ordinary differential equations, it is known that system equivalence can be translated into division relations among the polynomial matrices describing the system. In this paper it becomes apparent that in general the solution to the problem of system equivalence for AR-systems over rings explicitly depends on the module $M$ of all time-trajectories under consideration. Using the module $M$, the ring $Ω$ is extended to a ring $Ω,M$, and, under certain conditions on the ring