Coprime Factorizations of Multivariate Rational Matrices*

Eva Zerz

Abstract. Coprime factorization is a well-known issue in one-dimensional systems theory, having many applications in realization theory, balancing, controller synthesis, etc. Generalization to systems in more than one independent variable is a delicate matter. First, several nonequivalent coprimeness notions for multivariate polynomial matrices have been discussed in the literature: zero, minor, and factor coprimeness. Here we adopt a generalized version of factor primeness that appears to be most suitable for multidimensional systems: a matrix is prime if it is a minimal annihilator. After reformulating the shear concept of a factorization, it is shown that every rational matrix possesses left and right coprime factorizations that can be found by means of computer algebraic methods. Several properties of coprime factorizations are given in terms of certain determinantal ideals.

Key words. Coprime factorization, Multivariate polynomial matrices, Input–output relations of multidimensional systems, Generalized factor primeness, Minimal annihilators, Determinantal ideals.

1. Introduction

Matrix fraction descriptions of rational matrices are prominent in algebraic systems theory, see for instance [K1]. The idea is to express a rational matrix, usually interpreted as the transfer operator of a linear time-invariant system, as the “ratio” of two relatively prime polynomial matrices, just like in the scalar case, where any rational function can be represented as the quotient of two coprime polynomials (“pole-zero cancellation” in systems theory’s folklore).

Of course, commutativity is lost when passing from the scalar to the multi-variable case. Thus, for a rational transfer matrix $H$, left and right factorizations have to be distinguished:

$$H = D^{-1}N \quad \text{and} \quad H = ND^{-1},$$

the former corresponding to an input–output relation $Dy = Nu$, the latter to a driving variable description $u = \bar{D}v, \quad y = \bar{N}v$.


1 Department of Mathematics, University of Kaiserslautern, 67663 Kaiserslautern, Germany. zerz@mathematik.uni-kl.de.
An irreducible matrix fraction description, or a coprime factorization, is such that the numerator matrix \( N (\bar{N}) \) and the denominator matrix \( D (\bar{D}) \) are devoid of nontrivial common left (right) factors. The determinantal degree of such a (left or right) coprime factorization is minimal among the determinantal degrees of all possible factorizations of a rational matrix. This fact is crucial for realization theory.

The transfer matrices of multidimensional systems do not depend on one single variable (usually interpreted as frequency), but on several independent variables. The theory is far less developed. This is partially due to the annoying (or fascinating) diversity of multivariate primeness notions: for two-dimensional (2D) systems, zero and minor primeness already have to be distinguished, but the latter is still equivalent to factor primeness, see [MLK]. In dimensions greater than two, however, zero, minor, and factor primeness are nonequivalent notions, see [YG], [L], or [FV]. Zero and minor primeness can be formulated as requirements on the dimension of certain determinantal ideals. Wood et al. [WRO1] even introduced a so-called primeness degree, a quantity which measures the amount of primeness of a multivariate polynomial matrix, thus providing the “missing links” between zero and minor primeness.

On the other hand, a new factor primeness concept has been introduced by Oberst [O1], which is equivalent to torsion-freeness of a polynomial module associated to the system, and which has interesting interpretations in terms of controllability of multidimensional systems. The approach of the present paper is based on this primeness notion (“generalized factor primeness” [Z]).

After presenting the required concepts of multivariate factorizations in Section 2, coprimeness is characterized in Section 3, and a computational primeness test is given. The short but crucial Section 4 shows how to construct coprime factorizations of arbitrary multivariate rational matrices. Section 5 discusses some of the most important properties of coprime factorizations. Finally, Section 6 provides further insight into the structure of coprime factorizations, based on the theory of factorization ideals. Throughout the paper, similarities and differences between the uni-, bi-, and multivariate situations are stressed.

2. Notions

Throughout this paper, let \( \mathbb{F} \) be a field, and let \( \mathcal{A} \) be one of the following signal spaces: \( \mathcal{A} = C^\infty (\mathbb{R}^r) \) (with \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{F} = \mathbb{C} \)) in the continuous case, or \( \mathcal{A} = \mathbb{F}^N \) in the discrete case. The action of \( \mathcal{D} := \mathbb{F}[s_1, \ldots, s_r] \) on \( \mathcal{A} \) is given by partial derivation,

\[
s_i a = \hat{\partial}_i a \quad \text{for} \quad a \in C^\infty (\mathbb{R}^r), \quad 1 \leq i \leq r,
\]

or shifts,

\[
(s^m a)(n) = a(m + n) \quad \text{for} \quad a \in \mathbb{F}^N, \quad m, n \in \mathbb{N}^r,
\]

respectively. We make frequent use of Oberst’s “fundamental principle” [O2]:

A sequence

\[
\mathcal{D}^{1 \times q} \xrightarrow{R} \mathcal{G}^{1 \times q} M^{1 \times n} \mathcal{G}^{1 \times n}
\]