Differential Equations and Diophantine Approximation in Positive Characteristic

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Abstract. We prove that the Osgood-Thue Theorem, about Diophantine Approximation in function fields, holds under a more general condition when the ground field is finite.

1. Introduction

Let $K$ be a field of positive characteristic $p$, and let $K((T^{-1}))$ be the field of formal Laurent series. We consider the field of rational functions $K(T)$ as embedded into $K((T^{-1}))$. If $\alpha = \sum_{n=-\infty}^{n_0} a_n T^n$ is an element of $K((T^{-1}))$, with $a_{n_0} \neq 0$, the integer $n_0$ is the degree of $\alpha$, $n_0 = \deg \alpha$. We may put $\deg 0 = -\infty$, and we define an absolute value $|\cdot|$ on $K((T^{-1}))$ by $|\alpha| = q^{\deg \alpha}$ where $q > 1$. When $K$ is a finite field, we will take $q = |K|$. However, we will not restrict ourselves to this case.

It is well known that Roth’s Theorem fails in positive characteristic. Liouville’s Theorem holds, and a celebrated example of Mahler shows that this result is the best possible. However, it is possible to obtain more precise results while excluding some exceptional cases. For instance, it has been proved by OSGOOD ([3]) that $|\alpha - P/Q| \gg |Q|^{-[(n+3)/2]}$ if $\alpha$ is an algebraic element in $K((T^{-1}))$, of degree $n > 1$, which satisfies no rational Riccati differential equation (and $P$ and $Q$ are polynomials in $K[T], Q \neq 0$). Actually, the method of Osgood leads to $|\alpha - P/Q| \gg |Q|^{-[(n/2)+1]}$ (under the assumption that $\alpha$ satisfies no rational Riccati differential equation). It was recently proved by the authors ([2]) that if $\alpha$ satisfies no equation of the form $\alpha = (A\alpha^p + B)/(C\alpha^p + D)$, where $A, B, C$ and $D$ are coefficients in $K(T)$, not all zero, then $|\alpha - P/Q| \gg |Q|^{-[(n/2)+1+\varepsilon]}$ for every $\varepsilon > 0$. Our method was close to the original Thue’s method. Recently, a very interesting paper of VOLOCH ([6]) gave a result which allows us to adapt the method of Thue-Osgood: using this result, we can give another proof of our result when the ground field $K$ is finite. In this case, we can prove that the same result as...
the one given by Osgood, $|\alpha - P/Q| \gg |Q|^{-\left[\frac{n+1}{2}\right]}$, holds under the assumption that $\alpha$ satisfies no equation $\alpha = (A\alpha^p + B)/(C\alpha^p + D)$.

2. Osgood’s Theorem

In this section, we recall briefly the proof of Osgood’s Theorem, to make obvious the fact that Osgood’s method actually leads to the slightly improved result:

**Theorem 2.1.** (Osgood). Let $\alpha$ be an algebraic element in $K((T^{-1}))$, of degree $n > 1$ over $K(T)$. If $\alpha$ satisfies no rational Riccati differential equation, then $|\alpha - P/Q| \gg |Q|^{-\left[\frac{n+1}{2}\right]}$ for every pair $(P, Q)$ of elements of $K[T], Q \neq 0$.

**Proof.** First notice that every algebraic element $\alpha \in K((T^{-1}))$ is separable over $K(T)$, since it is clear that if $\alpha_1, \ldots, \alpha_n$ are elements of $K((T^{-1}))$ which are linearly independent over $K(T)$, then so are $\alpha_1^n, \ldots, \alpha_n^n$. Hence, $\alpha^n$ has the same degree $n$ as $\alpha$ over $K(T)$. As $\alpha$ is separable over $K(T)$, its derivative $\alpha'$ lies in $K(T)(\alpha)$: if $\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$, with coefficients $a_0, \ldots, a_{n-1}$, in $K(T)$, putting $F_1(X) = nx^n - \cdots + a_1$, and $F_2(X) = a_{n-1}X^{n-1} + \cdots + a_0$, we have $F_1(X) \neq 0$, hence $F_1(\alpha) \neq 0$, and since $\alpha^2 F_1(\alpha) + F_2(\alpha) = 0$, we get $\alpha' = -F_2(\alpha)/F_1(\alpha) \in K(T)(\alpha)$. Then the crucial point in the proof of Osgood’s Theorem is the use of Thue’s method in the following way: if $d$ is any integer with $0 \leq d < n$, the $n + 1$ elements $\alpha', \alpha^d, \ldots, \alpha^n$, are linearly dependent over $K(T)$, hence there exist polynomials $A(X)$ and $B(X)$ in $K[T][X]$, not both zero, such that $\deg_X A \leq n - d - 1$, $\deg_X B \leq d$, and $\alpha B(\alpha) = A(\alpha)$.

If $B(\alpha) = 0$, we have $A(\alpha) = 0$ and thus $A(X) = B(X) = 0$ since max $(\deg_X A, \deg_X B) < n$. As this is impossible, we thus have $B(\alpha) \neq 0$. Moreover, we can suppose that $A$ and $B$ are relatively prime. Consider the following “differential polynomial” on $K((T^{-1}))$: for each $\beta \in K((T^{-1}))$, put $H(\beta) = \beta \beta'B(\beta) - A(\beta)$. Notice that $|\beta' - \alpha'| \leq |\beta - \alpha|/|T|$. Moreover, when $|\beta - \alpha| \leq 1$, we have $|A(\beta) - A(\alpha)| \leq C_1|\alpha - \beta|$ and $|B(\beta) - B(\alpha)| \leq C_1|\alpha - \beta|$, where $C_1$ is a positive real constant ($C_1 = \max(\{|A|, |B|\}\max(1, |\alpha|^{n-2})$, where $|A|$ (respectively $|B|$) denotes the maximum of the absolute values of the coefficients of $A$ (resp. $B$), regarded as a polynomial with coefficients in $K[T]$). As $H(\alpha) = 0$, and $|B(\beta)| \leq \max(1, |\alpha|)C_1$, we thus have $|H(\beta)| \leq C_2|\alpha - \beta|$, with $C_2 = \max(1, |\alpha|)C_1$. Then let $P$ and $Q$ be elements of $K[T], Q \neq 0$. As $Q^2(P/Q)'$, as well as $Q^{n-d-1}A(P/Q)$ and $Q^d B(P/Q)$, are elements of $K[T]$, then $Q^\max(d+1, n-d-1)H(P/Q)$ lies in $K[T]$. Choosing $d = \lceil (n-2)/2 \rceil$, we have $d + 2 \geq n - d - 1$ since $\lceil (n-2)/2 \rceil \geq (n-3)/2$. Hence $\max(d+2, n-d-1) = d + 2 = \lceil n/2 \rceil + 1$. So if $H(P/Q) \neq 0$, we have $|H(P/Q)| \geq |Q|^{-\left[\frac{n+1}{2}\right]}$. Now it is proved in [3] or [4], that if the differential equation $H(\beta) = 0$ is not a Riccati equation, that is to say, if we have not both the conditions $\deg_X B = 0$ and $\deg_X A \leq 2$, then $H(P/Q) \neq 0$ for each pair $(P, Q)$ of coprime elements of $K[T]$, with $|Q|$ sufficiently large. So, for such $(P, Q)$, we get $|H(P/Q)| \geq |Q|^{-\left[\frac{n+1}{2}\right]}$, and thus $|\alpha - P/Q| \geq C_2^{-1}|Q|^{-\left[\frac{n+1}{2}\right]}$ (the condition $|\alpha - P/Q| \leq 1$ being removed since $C_2 \geq 1$).