Compactification of Limit $\mathcal{S}$-Net Spaces

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Abstract. Limit $\mathcal{S}$-net spaces are defined as convergence spaces whose convergence is expressed by using generalized nets, the so-called $\mathcal{S}$-nets (where $\mathcal{S}$ is a construct). For limit $\mathcal{S}$-net spaces we study compactifications, especially those ones that are analogous to the Alexandrov and Čech-Stone compactifications known for topological spaces.

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In the last decade, categories with closure operators have been intensively studied by many authors (see e.g. [2]) and at present they form an important subject of categorical topology. Categories of limit $\mathcal{S}$-net spaces were introduced and studied in [6] as special categories with closure operators. In [7] it was shown that in categories of limit $\mathcal{S}$-net spaces, compactness behaves more naturally than in generally viewed categories with closure operators, and even better than in the category of topological spaces. In the present paper we continue in investigating compactness of limit $\mathcal{S}$-net spaces by focussing our interest on their compactifications. We will show some similarities between behaviours of compactifications of limit $\mathcal{S}$-net spaces and those of topological spaces. Especially, we will find and study analogies of the Alexandrov and Čech-Stone compactifications of topological spaces.

For the categorical terminology used see [1]. Throughout the paper, all categories are considered to be constructs or, more generally, quasi-constructs, and $\mathcal{S}$ denotes a non-empty construct whose objects have non-empty underlying sets. As usual, we do not distinguish notationally between $\mathcal{S}$-objects, resp. $\mathcal{S}$-morphisms, and their underlying sets, resp. maps.

By an $\mathcal{S}$-net in a set $X$ we understand a pair $(S, f)$ where $S$ is an $\mathcal{S}$-object and $f : S \to X$ is a map. The class of all $\mathcal{S}$-nets in $X$ is denoted by $\langle X \rangle_\mathcal{S}$ and we accept the usual convention that $\langle X \rangle_\mathcal{S} \subseteq \langle Y \rangle_\mathcal{S}$ whenever $X \subseteq Y$. If $(S, f), (T, g) \in \langle X \rangle_\mathcal{S}$, then $(S, f)$ is called a subnet of $(T, g)$ provided that there is an $\mathcal{S}$-morphism $\varphi : S \to T$ such that $f = g \circ \varphi$. Note that the $\mathcal{S}$-nets in $X$ are nothing else than the

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objects of the comma category \((\mathcal{U} \downarrow X)\), i.e. the objects \(\mathcal{U}\)-over \(X\), where \(\mathcal{U} : \mathcal{S} \rightarrow \text{Set}\) is the underlying functor. Given a pair of \(\mathcal{S}\)-nets \((S, f), (T, g) \in \langle X \rangle_{\mathcal{S}}\), \((S, f)\) is a subnet of \((T, g)\) iff there is a morphism from \((S, f)\) into \((T, g)\) in \((\mathcal{U} \downarrow X)\).

Clearly, for \(\mathcal{S} = \text{Dir}\), where \(\text{Dir}\) denotes the category of directed sets and cofinal maps, the \(\mathcal{S}\)-nets and their subnets coincide with the usual nets and subnets.

Let \(X\) be a set and \(\pi : \langle X \rangle_{\mathcal{S}} \rightarrow \exp X\) a map. Then we write \((S, f) \xrightarrow{\pi} x\) and say that \((S, f)\) converges to \(x\) (with respect to \(\pi\)) if \(x \in \pi(S, f)\). An \(\mathcal{S}\)-net \((S, f)\) is said to be convergent (w.r.t. \(\pi\)) if there is a point \(x \in X\) with \((S, f) \xrightarrow{\pi} x\). We write \((S, f) \xrightarrow{\pi} x\) and say that \((S, f)\) does not converge to \(x\) (w.r.t. \(\pi\)) if \(x \in X - \pi(S, f)\).

**Definition 1.** Let \(X\) be a set and \(\pi : \langle X \rangle_{\mathcal{S}} \rightarrow \exp X\) a map. Then the pair \((X, \pi)\) is called a limit \(\mathcal{S}\)-net space if the following three conditions are fulfilled:

1. \((S, f) \xrightarrow{\pi} x\) whenever \(f : S \rightarrow X\) is the constant map with \(f(s) = x\) for all \(s \in S\).
2. If \((S, f) \xrightarrow{\pi} x\), then \((T, g) \xrightarrow{\pi} x\) for each subnet \((T, g)\) of \((S, f)\).
3. If \((S, f) \xrightarrow{\pi} x\), then there is a subnet \((T, g)\) of \((S, f)\) such that \((U, h) \xrightarrow{\pi} x\) for any subnet \((U, h)\) of \((T, g)\).

Let us note that in the case when \(\mathcal{S}\) is the category whose only object is the chain of natural numbers and whose morphisms are isotone injections, the limit \(\mathcal{S}\)-net spaces \((X, \pi)\) fulfilling card \(\pi(S, f) \leq 1\) for each \((S, f) \in \langle X \rangle_{\mathcal{S}}\) are nothing else than the well-known (Fréchet) \(\mathcal{S}^*\)-spaces (see e.g. [4]). In the case \(\mathcal{S} = \text{Dir}\) the limit \(\mathcal{S}\)-net spaces coincide with the \(L^*\)-spaces studied in [3].

We denote by \(\text{Lim}_{\mathcal{S}}\) the category of limit \(\mathcal{S}\)-net spaces and continuous maps, i.e. maps \(F : \langle X, \pi \rangle \rightarrow \langle Y, \rho \rangle\) fulfilling \((S, f) \xrightarrow{\pi} x \Rightarrow (S, F \circ f) \xrightarrow{\rho} F(x)\). The isomorphisms in \(\text{Lim}_{\mathcal{S}}\) are called homeomorphisms. By [6], \(\text{Lim}_{\mathcal{S}}\) is a topological category which is cartesian closed whenever \(\mathcal{S}\) has a concrete product for each pair of \(\mathcal{S}\)-objects.

Given a set \(X\), by a closure operation \(u\) on \(X\) we understand any map \(u : \exp X \rightarrow \exp X\) satisfying the following three axioms: \(u\emptyset = \emptyset\), \(A \subseteq X \Rightarrow A \subseteq uA\), and \(A \subseteq B \subseteq X \Rightarrow uA \subseteq uB\). The pair \((X, u)\) is then called a closure space. If \((X, u), (Y, v)\) are closure spaces, then a map \(F : (X, u) \rightarrow (Y, v)\) is said to be continuous if \(F(uA) \subseteq vF(A)\) for each \(A \subseteq X\). We denote by \(\text{Clo}\) the category of closure spaces and continuous maps. A closure operation \(u\) on \(X\) (and the closure space \((X, u)\)) is called additive if \(A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB\), and it is called idempotent if \(A \subseteq X \Rightarrow uuA = uA\). For each \(\mathcal{S}\)-net space \((X, \pi)\) and each subset \(A \subseteq X\) put \(u_A A = \{x \in X; \text{there is an } \mathcal{S}\text{-net } (S, f) \in \langle A \rangle_{\mathcal{S}} \text{ such that } (S, f) \xrightarrow{\pi} x\}\). By this way we have defined a concrete functor from \(\text{Lim}_{\mathcal{S}}\) to \(\text{Clo}\). We denote this functor by \(F_{\mathcal{S}}\).

**Definition 2.** Let \(\mathcal{S}\) have concrete products and let \((X, \pi)\) be a limit \(\mathcal{S}\)-net space. We say that \((X, \pi)\) fulfills CIL (i.e. the condition of iterated limits) if the following is valid:

Let \(S \in \mathcal{S}\) be an object and let \((T_s, g_s) \in \langle X \rangle_{\mathcal{S}}\) for each \(s \in S\). Let \(U = S \times \prod_{s \in S} T_s\) and let \(h : U \rightarrow X\) be the map given by \(h(s, t) = g_s(t(s))\). If \((T_s, g_s) \xrightarrow{\pi} x_s\) for each \(s \in S\) and \((S, f) \xrightarrow{\pi} x\), where \(f : S \rightarrow X\) is given by \(f(s) = x_s\), then \((U, h) \xrightarrow{\pi} x\).