CLASSICAL SOLUTIONS OF THE PERIODIC CAMASSA–HOLM EQUATION

G. MISIOLEK

Abstract
We study the periodic Cauchy problem for the Camassa–Holm equation and prove that it is locally well-posed in the space of continuously differentiable functions on the circle. The approach we use consists in rewriting the equation and deriving suitable estimates which permit application of o.d.e. techniques in Banach spaces. We also describe results in fractional Sobolev $H^s$ spaces and in Appendices provide a direct well-posedness proof for arbitrary real $s > 3/2$ based on commutator estimates of Kato and Ponce as well as include a derivation of the equation on the diffeomorphism group of the circle together with related curvature computations.

1 Introduction
In many respects the one-dimensional Camassa–Holm [CH] equation
\[ \partial_t u - \partial_t \partial_x^2 u + 3 u \partial_x u - 2 \partial_x u \partial_x^2 u - u \partial_x^3 u = 0 \quad (\text{CH}) \]
resembles the well-known Korteweg-de Vries equation
\[ \partial_t u + u \partial_x u + \partial_x^3 u = 0 . \]
Both equations are remarkable for their infinitely many first integrals, soliton-like solutions and bi-hamiltonian structures. They have the same symmetry group, the Bott–Virasoro group, where each can be viewed as a geodesic equation with respect to a different right-invariant metric. These and similar further results together with relevant background can be found in Fuchssteiner and Fokas [FF], Camassa and Holm [CH], [Mis1], Arnold and Khesin [AK], Beals, Sattinger and Szmigielski [BeSS1,2], Constantin and McKeen [ConM], McKean [Mc2] or Khesin and Misiolek [KhM] and references in these works.

It is useful to observe that the Camassa–Holm equation can be rewritten as
\[ \partial_t u + u \partial_x u + \Lambda^{-2} \partial_x \left( u^2 + \frac{1}{2} (\partial_x u)^2 \right) = 0 , \]
where $\Lambda^s = (1 - \partial_x^2)^{s/2}$. In fact, this is how it appears when derived directly on the Bott–Virasoro group and right-translated to the tangent space at the identity. Written in this form CH can be considered a nonlocal and nonlinear perturbation of the Burgers equation

$$\partial_t u + u \partial_x u = 0.$$ 

It is therefore not surprising that when it comes to well-posedness of the corresponding Cauchy problem CH has more in common with first order quasilinear equations such as Burgers equation rather than with KdV. Thus, it is known that for certain initial data the $L^\infty$ norm of the derivative of the corresponding solution becomes unbounded in finite time (this observation goes back to the original work of Camassa, Holm and Hyman [CH], [CHH] but see also subsequent papers of Constantin and Escher [ConE1,2] and particularly McKean [Mc1], where this problem is discussed in considerable detail). On the other hand, it can also be shown that if the initial data is real analytic then the resulting solution will be analytic in both variables, locally in time (see Himonas and Misiolek [HM2,3]).

In this paper we are interested in local well-posedness in the sense of Hadamard of classical solutions to the CH equation, that is existence, uniqueness and continuous dependence on the initial data of those solutions to the initial value problem

$$\partial_t u + u \partial_x u + \Lambda^{-2} \partial_x \left( u^2 + \frac{1}{2} (\partial_x u)^2 \right) = 0,$$  

$$u(x,0) = u_o(x), \quad x \in T \simeq \mathbb{R}/\mathbb{Z},$$

which are at least continuously differentiable and which satisfy equation (1.1) at every point of the domain.

As is well known, for quasilinear equations like the Burgers equation such solutions are obtained using for example the method of characteristics (see for instance John [Jo] or Hörmander [Hör]) or a variant of T. Kato’s abstract semigroup theory for nonreflexive Banach spaces (see Kato [K]). Even though these methods are not directly applicable to the CH equation we show in this paper that an analogous result does hold for the Cauchy problem stated above. More precisely, our main result is contained in the following theorem.

**Theorem 1.1.** Given any $u_o \in C^1(T)$ there exists a $T > 0$ and a unique solution $u$ to (1.1)–(1.2) such that $u \in C([0,T),C^1) \cap C^1([0,T),L^\infty)$ and for which the map $u_o \mapsto u$ is continuous in the $C^1$ norm.

This theorem improves all previously known results on local well-posedness of classical solutions to the periodic CH equation obtained for initial