Partial Parity \((g,f)-\text{Factors and Subgraphs Covering Given Vertex Subsets}\)

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Abstract. Let \(G\) be a graph and \(W\) a subset of \(V(G)\). Let \(g,f : V(G) \to \mathbb{Z}\) be two integer-valued functions such that \(g(x) \leq f(x)\) for all \(x \in V(G)\) and \(g(y) \equiv f(y) \pmod{2}\) for all \(y \in W\). Then a spanning subgraph \(F\) of \(G\) is called a partial parity \((g,f)\)-factor with respect to \(W\) if \(g(x) \leq \deg_F(x) \leq f(x)\) for all \(x \in V(G)\) and \(\deg_F(y) \equiv f(y) \pmod{2}\) for all \(y \in W\). We obtain a criterion for a graph \(G\) to have a partial parity \((g,f)\)-factor with respect to \(W\). Furthermore, by making use of this criterion, we give some necessary and sufficient conditions for a graph \(G\) to have a subgraph which covers \(W\) and has a certain given property.

1. Partial Parity \((g,f)\)-Factors

We consider a finite graph \(G\) which may have multiple edges and loops, and so a graph means such a graph throughout this paper. Let \(V(G)\) and \(E(G)\) denote the set of vertices and that of edges of \(G\), respectively. For two disjoint subsets \(S\) and \(T\) of \(V(G)\), we write \(e_G(S,T)\) for the number of edges of \(G\) joining \(S\) to \(T\). For a vertex \(v\) of \(G\), we denote by \(\deg_G(v)\) the degree of \(v\) in \(G\), and by \(N_G(v)\) the neighborhood of \(v\). Let \(\mathbb{Z}\) and \(\mathbb{Z}^+\) denote the set of integers and that of non-negative integers, respectively. For a function \(f : V(G) \to \mathbb{Z}^+\), a spanning subgraph \(F\) of \(G\) is called an \(f\)-factor if \(\deg_F(x) = f(x)\) for all \(x \in V(G)\). For two functions \(g,f : V(G) \to \mathbb{Z}\) such that \(g(x) \leq f(x)\) for all \(x \in V(G)\), a spanning subgraph \(H\) of \(G\) is called a \((g,f)\)-factor if \(g(x) \leq \deg_H(x) \leq f(x)\) for all \(x \in V(G)\). Note that when we consider \((g,f)\)-factors, we allow that \(g(x) < 0\) and/or \(\deg_G(y) < f(y)\) for some vertices \(x\) and \(y\) of \(G\), and this relaxation will play an important technical role.

For a given subset \(W\) of \(V(G)\), let \(g,f : V(G) \to \mathbb{Z}\) be two functions such that \(g(x) \leq f(x)\) for all \(x \in V(G)\) and \(g(y) \equiv f(y) \pmod{2}\) for all \(y \in W\). Then a spanning subgraph \(F\) of \(G\) is called a partial parity \((g,f)\)-factor with respect to \(W\) if
\[ g(x) \leq \deg_F(x) \leq f(x) \quad \text{for all } x \in V(G) \quad \text{and} \]
\[ \deg_F(y) \equiv g(y) \equiv f(y) \pmod{2} \quad \text{for all } y \in W. \]

Note that if \( W = \emptyset \), then a partial parity \((g,f)\)-factor is a \((g,f)\)-factor and, if \( W = V(G) \), then a partial parity \((g,f)\)-factor is briefly called a parity \((g,f)\)-factor. The criterions for a graph to have an \( f \)-factor, a \((g,f)\)-factor and a parity \((g,f)\)-factor were obtained by Tutte [9], Lovász [4] and [6], respectively. Moreover, Niessen [7] recently gave a criterion for a graph \( G \) to have all \((g,f)\)-factors, where we say that \( G \) has all \((g,f)\)-factors if \( G \) has an \( h \)-factor for every \( h : V(G) \to \mathbb{Z}^+ \) such that \( g(x) \leq h(x) \leq f(x) \) for all \( x \in V(G) \) and \( \sum_{x \in V(G)} h(x) \equiv 0 \pmod{2} \), and if at least one such \( h \) exists.

We first give a necessary and sufficient condition for a graph to have a partial parity \((g,f)\)-factor.

**Theorem 1.** Let \( G \) be a graph and \( W \) a subset of \( V(G) \). Let \( g, f : V(G) \to \mathbb{Z} \) be two functions satisfying
\[ g(x) \leq f(x) \quad \text{for all } x \in V(G), \quad \text{and} \quad g(y) \equiv f(y) \pmod{2} \quad \text{for all } y \in W. \]

Then \( G \) has a partial parity \((g,f)\)-factor with respect to \( W \) if and only if for all disjoint subsets \( S \) and \( T \) of \( V(G) \),
\[ \eta_G(S, T) = \sum_{x \in S} f(x) + \sum_{x \in T} (\deg_G(x) - g(x)) - e_G(S, T) - k_W(S, T) \geq 0, \quad (1) \]

where \( k_W(S, T) \) denotes the number of components \( D \) of \( G - (S \cup T) \) such that
\[ g(x) = f(x) \quad \text{for all } x \in V(D) \setminus W \text{ and } \sum_{x \in V(D)} f(x) + e_G(V(D), T) \equiv 1 \pmod{2}. \quad (2) \]

In order to prove Theorem 1, we need the following \((g,f)\)-factor theorem.

**Theorem 2 (Lovász [4]).** Let \( G \) be a graph and \( g, f : V(G) \to \mathbb{Z} \). Then \( G \) has a \((g,f)\)-factor if and only if for all disjoint subsets \( S \) and \( T \) of \( V(G) \),
\[ \delta_G(S, T) = \sum_{x \in S} f(x) + \sum_{x \in T} (\deg_G(x) - g(x)) - e_G(S, T) - h_G(S, T) \geq 0, \quad (3) \]

where \( h_G(S, T) \) denotes the number of components \( D \) of \( G - (S \cup T) \) such that
\[ g(x) = f(x) \text{ for all } x \in V(D) \text{ and } \sum_{x \in V(D)} f(x) + e_G(V(D), T) \equiv 1 \pmod{2}. \quad (4) \]