A Property About Minimum Edge- and Minimum Clique-Cover of a Graph

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Abstract. Let $G$ be a graph with $n$ vertices, and denote as $\gamma(G)$ (as $\theta(G)$) the cardinality of a minimum edge cover (of a minimum clique cover) of $G$. Let $E$ (let $C$) be the edge-vertex (the clique-vertex) incidence matrix of $G$; write then $P(E) = \{x \in \mathbb{R}^n : Ex \leq 1, x \geq 0\}$, $P(C) = \{x \in \mathbb{R}^n : Cx \leq 1, x \geq 0\}$, $z_E(G) = \max\{1^T x \text{ subject to } x \in P(E)\}$, and $z_C(G) = \max\{1^T x \text{ subject to } x \in P(C)\}$. In this paper we prove that if $z_E(G) = z_C(G)$, then $\gamma(G) = \theta(G)$.

1. Introduction

The graphs we consider are connected, undirected and with no loops. Let $G = (V, E)$ be a graph, with $|V| = n$. For any $F \subseteq E$, denote as $G(F)$ the subgraph of $G$ induced by $F$. For any $W \subseteq V$, denote as $G[W]$ the subgraph of $G$ induced by $W$. Denote then as $N(W)$ the set of vertices of $V \setminus W$ that are adjacent to some vertex of $W$; if $W$ is a singleton, i.e., $W = \{w\}$, then we simply write $N(w)$ instead of $N(\{w\})$. Two edges (two vertices) are independent if they do not share any vertex (any edge). A matching (a stable) of $G$ is a set of pairwise independent edges (vertices) of $G$. A matching $M$ (a stable $S$) of $G$ is maximum if $|M| \geq |M'|$ (if $|S| \geq |S'|$) for any matching $M'$ (for any stable $S'$) of $G$. Denote as $\mu(G)$ (as $\alpha(G)$) the cardinality of a maximum matching (stable) of $G$. A clique of $G$ is a set of mutually adjacent vertices of $G$. We say that a clique $K$ is a big clique if $|K| \geq 3$. An edge cover (a clique cover) of $G$ is a set $F$ of edges (a set $Q$ of cliques) of $G$ such that each vertex of $G$ belongs to some element of $F$ (of $Q$). An edge cover $F$ (a clique cover $Q$) of $G$ is minimum if $|F| \leq |F'|$ (if $|Q| \leq |Q'|$) for any edge cover $F'$ (for any clique cover $Q'$) of $G$. Denote as $\gamma(G)$ (as $\theta(G)$) the cardinality of a minimum edge cover (of a minimum clique cover) of $G$.

Let $M$ be a matching of $G$; denote as $V(M)$ the set of vertices of $G$ belonging to an edge of $M$. Vertices in $V(M)$ are said to be saturated by $M$. Furthermore, given a path $P = v_0, v_1, \ldots, v_k$ in $G$, $P$ is alternating with respect to $M$ if each edge in $P$ that does not belong to $M$ is followed in $P$ by an edge that belongs to $M$, and vice-
versa. If $P$ is alternating, index $k$ is odd, and $v_0, v_k \notin V(M)$, then $P$ is augmenting with respect to $M$. Given two sets $M'$ and $M''$, then let us write $M' \oplus M'' = (M' \setminus M'') \cup (M'' \setminus M')$. If $M'$ and $M''$ are matchings, then the components of $G(M' \oplus M'')$ are clearly paths and even cycles of $G$; denote such paths as path-components of $G(M' \oplus M'')$. Finally $G$ admits a perfect matching if it contains a matching $M$ such that $V(M) = V$.

An edge-vertex (or clique-vertex) matrix is a matrix $E$ (a matrix $C$) whose rows are associated with the edges (with the maximal cliques) of $G$, and whose columns are associated with the vertices of $G$: entry $E_{ij}$ (entry $C_{ij}$) is 1 if the $i$-th edge (if the $i$-th clique) contains the $j$-th vertex, and 0 otherwise.

The maximum stable set problem (MS) is that of computing a subset $S$ of pairwise non-adjacent vertices of $G$ having maximum cardinality. The cardinality of $S$ is usually denoted as $\alpha(G)$. It is well known that MS is NP-hard (see e. g. [8]). If one associates with each vertex $i$ of $G$ a variable $x_i$, and considers polytopes $P(E) = \{x \in \mathbb{R}^n : Ex \leq 1, x \geq 0\}$, and $P(C) = \{x \in \mathbb{R}^n : Cx \leq 1, x \geq 0\}$, then both the following linear programs:

\[
\begin{align*}
\text{max} & \quad 1^T x \\
\text{subject to} & \quad x \in P(E),
\end{align*}
\]

and

\[
\begin{align*}
\text{max} & \quad 1^T x \\
\text{subject to} & \quad x \in P(C),
\end{align*}
\]

are expressions of $\alpha(G)$, relaxed with respect to binary constraints. Denote as $\alpha_E(G)$ and $\alpha_C(G)$ the optimal values of (1) and (2) respectively, and as $\text{STAB}(G)$ the convex-hull of the incidence vectors of all stable sets in $G$: the relation $\text{STAB}(G) \subseteq P(C) \subseteq P(E)$ implies that $\alpha(G) \leq \alpha_C(G) \leq \alpha_E(G)$. It is well known that a graph $G$ with no isolated vertices is bipartite if and only if $\text{STAB}(G) = P(E)$; furthermore Chvátal proved that a graph $G$ is perfect if and only if $\text{STAB}(G) = P(C)$ (see e. g. [4] for both).

In [1], among several results, it is proved that $\alpha(G) = \gamma(G)$ if and only if $\alpha(G) = \alpha_E(G)$ (a connected graph $G$ is usually said Kőnig if $\alpha(G) = \gamma(G)$). One can focus this result by considering the dual problems of (1) and (2):

\[
\begin{align*}
\text{min} & \quad 1^T y \\
\text{subject to} & \quad y \in \{y \in \mathbb{R}^n : y^T E \geq 1, y \geq 0\},
\end{align*}
\]

and

\[
\begin{align*}
\text{min} & \quad 1^T y \\
\text{subject to} & \quad y \in \{y \in \mathbb{R}^n : y^T C \geq 1, y \geq 0\},
\end{align*}
\]

which are respectively an expression of $\gamma(G)$ and of $\theta(G)$, relaxed with respect to binary constraints. These aspects can lead to wonder if there exist further non-immediate relations among values $\alpha(G)$, $\alpha_E(G)$, $\alpha_C(G)$, $\gamma(G)$ and $\theta(G)$.

In this paper we consider graphs $G$ for which equality

\[\alpha_E(G) = \alpha_C(G).\]