We discuss the string picture behind the integrable spin chains governing the evolution equations in the Yang–Mills theory. We show that the one-loop correction to the dilatation operator in the $N=4$ theory can be expressed in terms of two-point correlation functions on the two-dimensional worldsheet. Using the relation between the Neumann integrable system and spin chains, we argue that the transition to the finite gauge-theory coupling implies discretization of the worldsheet. We conjecture that the string-bit model for the discretized worldsheet corresponds to the representation of the integrable spin chains in terms of the separated variables.

Keywords: spin chains, duality, string theory

1. Introduction

The explicit realization of the gauge–string duality has remained a challenging problem during the decades since the early formulations [1]. The important result [2] in the $N=4$ supersymmetric Yang–Mills (SYM) case is the direct mapping between the modes of the closed string in the $AdS_5 \times S_5$ background and the operators in the gauge theory [3]. The anomalous dimensions of the gauge-theory operators are identified with the energies of the corresponding string excitations. But the problem remains far from being solved, even in the most symmetric case of the $N=4$ supersymmetry. The key problem is the lack of a prediction for the anomalous dimensions of the gauge-theory operators with a finite coupling or, equivalently, the lack of a proper formulation of the string picture for the weak coupling in the gauge theory. With the strong coupling, the semiclassical Nambu–Goto string picture in an AdS-type background is valid, which, for instance, yields explicit predictions for the Wilson loop observables [4]. The problem becomes more involved with the weak coupling because the string must then be considered quantum mechanically, and the whole spectrum of the string modes must be taken into account. Hence, the intermediate goal is to identify the closed-string dynamics relevant for both the weak and strong coupling regimes and to find the complete mapping between generic multiparticle conformal operators and the string modes in the framework of the gauge–string duality.

Another approach for calculating the anomalous dimensions of the gauge-theory operators includes open strings. It involves calculating the Wilson loop observables for some specific contours. The Wilson loop approach in quantum chromodynamics was formulated in [1] and was recently revisited in [5] in relation to the underlying string picture. Calculations of the Wilson loops with a strong coupling precisely match the closed-string calculations [6], and the mapping to the topological theory was used in [5] to argue that the relevant string with the weak coupling seems to be tensionless in agreement with the earlier expectation. In particular, the cusp anomaly, which is a generating function for the anomalous dimensions of the large-spin twist-two light-cone operators [7], was related in the weak-coupling regime to the disc amplitude in the
two-dimensional Yang–Mills (YM) theory admitting the string representation.

In general, operators mix during the renormalization-group (RG) evolution, and to find anomalous dimensions of the multiplicatively renormalized operators, the dilatation operator, which can be represented as an infinite- or finite-dimensional matrix in the Hilbert space of the gauge theory, must be diagonalized. It appears that the dilatation operator found in [8] at the one-loop level coincides with the Hamiltonian of a finite-dimensional integrable system. Actually, the phenomenon of the hidden integrability of the RG flows at large $N$ seems to be quite general, at least at the one-loop level. It has been observed for low-twist light-cone operators [9] and arbitrary twist quasipartonic operators in the $SL(2, \mathbb{R})$ sector [5], for operators with a large $R$ charge in the $N=4$ SYM theory [10], and also for evolution in the Regge limit [11]. Precisely, it appeared that at the one-loop level in these cases, the dilatation operators can be identified with the respective Hamiltonians of the $SL(2, \mathbb{R})$, $SO(6)$, and $SL(2, \mathbb{C})$ spin chains. The unifying integrable spin chain on the $SU(2, 2|4)$ supergroup responsible for the evolution of $N=4$ supersymmetric operators was recently formulated [12]. The integrability of the many-body system implies the existence of higher integrals of motion, which are in involution with the dilatation operator. Hence, the anomalous dimension of the operator involving $N$ constituents generically depends on $N-2$ hidden quantum numbers. Another important issue concerning the role of nonlocal integrals of motion was recently raised in [13] (also see [14] for an earlier discussion of this point).

The proper place for these integrable systems must be found in the gauge–string duality approach. It was argued in [15] that the hidden integrability could be the key property for the string description of the weak-coupling limit in the gauge theory. Moreover, the more general features of integrable systems, such as the existence of a huge number of local and nonlocal integrals of motion, the appearance of the Riemann surfaces as the spectral curves, the possibility of separating the variables, etc., must be reflected in the string picture. A natural hypothesis of where an integrable finite-dimensional system can arise is connected with discretization of the worldsheet. Such an idea was discussed long ago in the context of the string-bit model in the light-cone gauge [16].

In this paper, we seek the string origin of these integrable systems. First, we consider various representations of the cusp anomaly yielding the anomalous dimensions of the light-cone operators with a large $SL(2, \mathbb{R})$ spin. It was argued in [5] that this anomaly can be represented in terms of the propagator of the particle on the $SL(2, \mathbb{R})$ group manifold or as the disc partition function of the $SL(2, \mathbb{R})$ two-dimensional YM theory. We show that there exists one more representation provided the worldsheet description involves the two-point correlation function in the theory of the massive free two-dimensional scalar particle. Because the cusp anomaly provides the anomalous dimensions only for large-spin operators, we also seek a similar two-point representation in the two-dimensional theory for arbitrary spin operators. We show that such a representation can be obtained from the same scalar theory after rotation to the Rindler Hilbert space in the worldsheet theory. Another argument involves the generalization of the particle motion on the $SL(2, \mathbb{R})$ manifold (or, equivalently, on the AdS$_3$ manifold) to the string motion on the same manifold.

In addition to deriving anomalous dimensions using the cusp anomaly, we also try to identify the origin of the spin chains. The natural object arising in the description of the spin chains governing the evolution of light-cone operators appears to be the exactly solvable (2+1)-dimensional gravity [17]. In the negative cosmological constant case, it is described by the $SL(2, \mathbb{C})$ Chern–Simons (CS) theory, while in the flat space, the description is given via the $ISO(2,1)$ gauge group. Moduli of the flat connections representing the phase space of the CS theory can be mapped into the moduli of the complex structures of the fixed-time surfaces in the Hamiltonian description of three-dimensional gravity. To incorporate spin chains, we must add Wilson lines representing point particles and consider the moduli space of flat connections on the surface with punctures [18]. The monodromies of the connection around the marked points can then be related to the local Lax operators of the spin chains. We note that this relation between statistical models and