Formulas for the Power Series Coefficients of Certain Quotients of Eisenstein Series

11.1 Introduction

In their epic paper [176], [242, pp. 276–309], G.H. Hardy and S. Ramanujan found an asymptotic formula for the partition function \( p(n) \) that arises from the power series coefficients of the reciprocal of the Dedekind eta function. As they indicated near the end of their paper, their methods also apply to several analogues of the partition function generated by modular forms of negative weight that are analytic in the upper half-plane. In their last published paper [177], [242, pp. 310–321], they considered a similar problem for the coefficients of modular forms of negative weight having a simple pole in a fundamental region, and in particular, they applied their theorem to find interesting series representations for the coefficients of the reciprocal of the Eisenstein series \( E_6(\tau) \). Although there are some similarities in the methods of these papers, the principal ideas are quite different in [177] from those in [176]. In [176], Hardy and Ramanujan introduced their famous circle method, and since that time the ideas in this paper have had an enormous impact in additive analytic number theory. Although their paper [177] has not had as much influence, the ideas in [177] have been extended by, among others, J. Lehner [201], H. Petersson [228], [229], [230], H. Poincaré [231], and H.S. Zuckerman [291]. Additional comments on [177] can be found in the third edition of [242, p. 387].

While confined to nursing homes and sanitariums during his last two years in England, Ramanujan wrote several letters to Hardy about the coefficients in the power series expansions of certain quotients of Eisenstein series. A few pages in his lost notebook are also devoted to this topic. All of this material can be found in [244, pp. 97–126], and the letters with commentary can be found in the book by Berndt and R.A. Rankin [74, pp. 175–191]. In these letters and in the lost notebook, Ramanujan claims formulas for the coefficients of several quotients of Eisenstein series not examined by Hardy and him in [177]. In fact, for some of these quotients, the main theorem of [177] needs to be moderately modified and improved. For other examples, a significantly stronger theorem is necessary. Ramanujan obviously wanted another exam-
ple to be included in their paper [177], for in his letter of 28 June 1918 [74, pp. 182–183], he wrote, “I am sending you the analogous results in case of $g_2$. Please mention them in the paper without proof. After all we have got only two neat examples to offer, viz. $g_2$ and $g_3$. So please don’t omit the results.” This letter was evidently written after galley proofs for [177] were printed, because Ramanujan’s request went unheeded. The functions $g_2$ and $g_3$, defined in (9.2.2) and (9.2.3), respectively, are the familiar invariants in the theory of elliptic functions and are constant multiples of the Eisenstein series $E_4(\tau)$ and $E_6(\tau)$, respectively. This letter was also evidently written before Ramanujan obtained further examples.

In this chapter, we establish the formulas for the coefficients of those quotients of Eisenstein series found in [244, pp. 102–104, 117]. In Ramanujan's notation, the three relevant Eisenstein series are defined, for $|q| < 1$, by

$$
P(q) := 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k}, \quad (11.1.1)$$

$$
Q(q) := 1 + 240 \sum_{k=1}^{\infty} \frac{k^3q^k}{1-q^k}, \quad (11.1.2)$$

and

$$
R(q) := 1 - 504 \sum_{k=1}^{\infty} \frac{k^5q^k}{1-q^k}. \quad (11.1.3)$$

(The notation above is that used in Ramanujan’s paper [240], [242, pp. 136–162] and in his lost notebook [244]. In his notebooks [243], Ramanujan replaced $P, Q,$ and $R$ by $L, M,$ and $N,$ respectively.) In more contemporary notation, the Eisenstein series $E_{2j}(\tau)$ is defined for $j > 1$ and $\text{Im} \ \tau > 0$ by

$$
E_{2j}(\tau) := \frac{1}{2} \sum_{(m_1, m_2) \in \mathbb{Z}} (m_1 \tau + m_2)^{-2j} = 1 - \frac{4j}{B_{2j}} \sum_{k=1}^{\infty} \frac{k^{2j-1}q^k}{1-q^k} = 1 - \frac{4j}{B_{2j}} \sum_{r=1}^{\infty} \sigma_{2j-1}(r)q^r, \quad (11.1.4)
$$

where $q = e^{2\pi i \tau}$, $B_j, j \geq 0$, denotes the $j$th Bernoulli number, and $\sigma_\nu(n) = \sum_{d|n} d^\nu$. Thus, for $q = \exp(2\pi i \tau)$, $E_4(\tau) = Q(q)$ and $E_6(\tau) = R(q)$, which have weights 4 and 6, respectively [255, p. 50]. Since (11.1.4) does not converge for $j = 1$, the Eisenstein series $E_2(\tau)$ must be defined differently. First let

$$
E_2^*(\tau) := P(q), \quad q = e^{2\pi i \tau}. \quad (11.1.5)
$$

Then $E_2(\tau)$ is defined by