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Eisenstein Series and Modular Equations

13.1 Introduction

Recall that Ramanujan’s three Eisenstein series $P(q)$, $Q(q)$, and $R(q)$ are defined by

\[ P(q) := 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1 - q^k}, \]  
(13.1.1)

\[ Q(q) := 1 + 240 \sum_{k=1}^{\infty} \frac{k^3q^k}{1 - q^k}, \]  
(13.1.2)

and

\[ R(q) := 1 - 504 \sum_{k=1}^{\infty} \frac{k^5q^k}{1 - q^k}, \]  
(13.1.3)

where $|q| < 1$. On pages 44, 50, 51, and 53 in his lost notebook [244], Ramanujan offers 12 formulas for Eisenstein series. All are connected with modular equations of degree either 5 or 7.

In a wonderful paper [233] devoted to proving identities for Eisenstein series and incomplete elliptic integrals in Ramanujan’s lost notebook, S. Raghavan and S.S. Rangachari employ the theory of modular forms in establishing proofs for all of Ramanujan’s identities for Eisenstein series. Most of the identities give representations for certain Eisenstein series in terms of quotients of Dedekind eta functions, or, more precisely, Hauptmoduls. The very short proofs by Raghavan and Rangachari depend on the finite dimensions of the spaces of relevant modular forms, and therefore upon showing that a sufficient number of coefficients in the expansions about $q = 0$ of both sides of the proposed identities agree. Ramanujan evidently was unfamiliar with the theory of modular forms and most likely did not discover the identities by comparing coefficients.

The purpose of this chapter is therefore to construct proofs in the spirit of Ramanujan’s work. In fact, our proofs depend only on theorems found in Ramanujan’s notebooks [243]. Admittedly, some of our algebraic manipulations are rather laborious, and we resorted at times to Mathematica. It is therefore clear to us that Ramanujan’s calculations, at least in some cases, were more elegant than ours. We actually have devised two approaches. In Sections 13.3 and 13.4, we use the two methods, respectively, to prove Ramanujan’s quintic identities. At the end of Section 13.3, we prove a first-order nonlinear “quintic” differential equation of Ramanujan satisfied by $P(q)$. In Section 13.5, we use the second approach, which is more constructive, to prove Ramanujan’s septic identities. The new parameterizations for moduli of degree 7 in Section 13.5 appear to be more useful than those given in [54, Section 19]. In Section 13.6, we briefly describe two new first-order nonlinear “septic” differential equations for $P(q)$.

The content of this chapter is taken from a paper by Berndt, H.H. Chan, J. Sohn, and S.H. Son [67].

13.2 Preliminary Results

Define, after Ramanujan,

$$f(-q) := (q; q)_{\infty} =: e^{-2\pi i z/24} \eta(z), \quad q = e^{2\pi i z}, \quad \text{Im } z > 0,$$

(13.2.1)

where $\eta$ denotes the Dedekind eta function. We shall use the well-known transformation formula [54, p. 43, Entry 27(iii)]

$$\eta(-1/z) = \sqrt{z/i} \eta(z).$$

(13.2.2)

The Eisenstein series $Q(q)$ and $R(q)$ are modular forms of weights 4 and 6, respectively. In particular, they obey the easily proved and well-known transformation formulas [246, p. 136]

$$Q(e^{-2\pi i z}) = z^4 Q(e^{2\pi i z})$$

(13.2.3)

and

$$R(e^{-2\pi i z}) = z^6 R(e^{2\pi i z}).$$

(13.2.4)

Our proofs below depend on modular equations. As usual, set

$$(a)_k := \frac{\Gamma(a + k)}{\Gamma(a)}$$

and

$$2F_1(a, b; c; x) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k, \quad |x| < 1.$$

Suppose that, for some positive integer $n$, 