Appendix B
Compatible Finite Element Spaces

This appendix is intended to serve as a quick reference guide to the key finite element spaces used in this book and their basic properties. Its content is mostly devoted to finite element approximation of the De Rham complex (A.52), i.e., finite element spaces \( G^h, C^h, D^h, \) and \( S^h \) that, in addition to being proper subspaces of \( G(\Omega), C(\Omega), D(\Omega), \) and \( S(\Omega) \), respectively, also form an exact sequence.

The chief reason to consider simultaneous approximation of \( G(\Omega), C(\Omega), D(\Omega), \) and \( S(\Omega) \), instead of dealing with them one by one, is the recent appreciation of the fact that the stability and/or accuracy of many finite element methods,\(^1\) including some least-squares finite element methods (LSFEMs), are contingent upon the existence of a discrete De Rham complex \( \{G^h(\Omega), C^h(\Omega), D^h(\Omega), S^h(\Omega)\} \) and bounded projection operators \( \Pi \) such that for \( \beta \in \{\emptyset, \gamma\} \) the diagram

\[
\begin{align*}
G^\beta(\Omega) \xrightarrow{\nabla} C^\beta(\Omega) & \xrightarrow{\nabla \times} D^\beta(\Omega) & \xrightarrow{\nabla \cdot} S^\beta(\Omega) \\
\Pi_G \downarrow & \quad \Pi_C \downarrow & \quad \Pi_D \downarrow & \quad \Pi_S \downarrow \\
G^h_\beta(\Omega) \xrightarrow{\nabla} C^h_\beta(\Omega) & \xrightarrow{\nabla \times} D^h_\beta(\Omega) & \xrightarrow{\nabla \cdot} S^h_\beta(\Omega)
\end{align*}
\]

(B.1)

commutes; see, e.g., [7, 69, 73, 137, 204, 206]. In what follows, we refer to finite element spaces that form a discrete De Rham complex as \textit{compatible} approximations of \( G(\Omega), C(\Omega), D(\Omega), \) and \( S(\Omega) \), or simply as \textit{compatible finite element spaces}.

In the finite element literature, the term “standard” or “nodal” finite elements is often used to describe approximations of the Sobolev space \( G(\Omega) = H^1(\Omega) \). Because \( H^1(\Omega) \) coincides with the first space in the De Rham complex, its approximations are provided by \( G^h(\Omega) \) which obviates the need for a separate discussion of standard spaces.

\(^1\) The same realization emerged at about the same time and independently in finite difference methods [139, 224, 314] and finite volume methods [276, 287, 289] which further underscores the relevance of this viewpoint in numerical methods for partial differential equations. More information about the use of homological ideas in discretizations is found in, e.g., [8, 13, 40, 75, 87, 121, 137, 204].

B.1 Formal Definition and Properties of Finite Element Spaces

A finite element in $\mathbb{R}^d$, $d = 1, 2, 3$, (see [123, p.78]) is a triple $\{\kappa, P, \Lambda\}$, where

1. $\kappa$ is a closed subset of $\mathbb{R}^d$ with nonempty interior and a Lipschitz-continuous boundary
2. $P(\kappa)$ is an $n$-dimensional space of real scalar or vector-valued functions defined over $\kappa$
3. $\Lambda(\kappa)$ is a unisolvent set of $n$ linear functionals $l_i : P(\kappa) \to \mathbb{R}$, i.e., $l_i$ are linearly independent and, for every set $\{a_1, \ldots, a_n\}$ of $n$ real numbers, there exists a unique $u \in P(\kappa)$ such that $l_i(u) = a_i$.

The set $\Lambda$ constitutes the degrees of freedom of the element $\{\kappa, P, \Lambda\}$. The functions $\{u_i\}_{i=1}^n$ such that $l_i(u_j) = \delta_{ij}$ form a basis for $P(\kappa)$ that is dual to $\{l_i\}_{i=1}^n$. Let $X(\kappa)$ be a space of smooth functions so that $l_i(v)$ are defined for all $v \in X(\kappa)$ and $P(\kappa) \subset X(\kappa)$. The local canonical projection operator $X(\kappa) \to P(\kappa)$ is defined by the formula

$$
\Pi_\kappa(v) = \sum_{i=1}^n l_i(v)u_i.
$$

A key feature of finite element methods is that every finite element $\{\kappa, P, \Lambda\}$ is completely defined by its reference element $\{\hat{\kappa}, \hat{P}, \hat{\Lambda}\}$ and a diffeomorphism $\hat{F}_\kappa : \hat{\kappa} \to \kappa$. As a result, the shape of $\kappa$ must be such that one can easily construct the diffeomorphism $F_\kappa$. For this reason, reference regions $\hat{\kappa}$ are drawn from a relatively small set of standard shapes for which there exist polynomial diffeomorphisms $F_\kappa$. Examples include 3-simplices (tetrahedrons), 3-cubes (hexahedrons), and prisms in three dimensions and 2-simplices (triangles) and 2-cubes (quadrilaterals) in two dimensions.\(^2\)

If $F_\kappa$ and $\hat{P}(\hat{\kappa})$ are defined using polynomials of the same degree, the element $\{\kappa, P, \Lambda\}$ is called isoparametric; see [123, Remark 4.3.1, p. 226]. A set $\{\{\kappa, P, \Lambda\}\}$ of finite elements is referred to as an affine family if $F_\kappa(\hat{\kappa}) = A_\kappa \hat{x} + \vec{b}_\kappa$ for every element in the set; here, the $A_\kappa$s and $\vec{b}_\kappa$s are $d \times d$ matrices and $d$ vectors, respectively. In all other cases, $\{\{\kappa, P, \Lambda\}\}$ is called a non-affine family.

Given a reference element $\{\hat{\kappa}, \hat{P}, \hat{\Lambda}\}$ with a basis $\{\hat{u}_i\}$, the basis $\{u_i\}$ of $\{\kappa, P, \Lambda\}$ is defined by pullback:\(^4\)

$$
u_i = \Phi^*(\hat{u}_i), \quad i = 1, \ldots, n.
$$

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\(^2\) Recall that $F_\kappa$ is a diffeomorphism if it is a bijective map such that $F_\kappa$ and $F_\kappa^{-1}$ are differentiable.

\(^3\) In principle, $\hat{P}(\hat{\kappa})$ can be any finite dimensional space such as the first $n$ Fourier modes, a finite set of wavelet functions, and so on. In practice, the prevalent choice for finite element methods has been to use polynomial functions. This choice combines simplicity with good approximation properties and leads to the easy determination of basis functions. For this reason, we restrict attention to polynomial finite element spaces.

\(^4\) The actual form of $\Phi^*$ is discussed in Section B.2 and depends on which one of the four function spaces $G(\Omega)$, $C(\Omega)$, $D(\Omega)$, or $S(\Omega)$ is being approximated.