Chapter IV
The Cauchy Problem (Continued)

In this chapter, under the assumption that the quasilinear hyperbolic system is not weakly linearly degenerate (WLD), we consider the formation of singularities of the $C^1$ solution to the Cauchy problem and its blow-up mechanism.

4.1 Some Uniform a Priori Estimates Independent of Weak Linear Degeneracy

In this section we give some uniform a priori estimates independent of weak linear degeneracy for the following Cauchy problem:

\[
\begin{align*}
\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} &= 0, \\
t = 0: \ u = \phi(x),
\end{align*}
\]

(4.1.1)
(4.1.2)

where (4.1.1) is a strictly hyperbolic system and $\phi(x)$ satisfies (3.2.1). We already proved Lemmas 3.2.1 and 3.2.2 in Section 3.2.

In order to obtain a sharp estimate on the life span of the $C^1$ solution to the Cauchy problem, we assume that the initial data are of the form

\[
\phi(x) = \varepsilon \psi(x),
\]

(4.1.3)

where $\varepsilon > 0$ is a small parameter and $\psi(x) \in C^1$ satisfies

\[
\sup_{x \in \mathbb{R}} \{(1 + |x|)^{1+\mu}(|\psi(x)| + |\psi'(x)|)\} < +\infty,
\]

(4.1.4)

where $\mu > 0$ is a constant.

In this situation we rewrite Lemma 3.2.2 as

**Lemma 4.1.1** Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$ and system (4.1.1) is strictly hyperbolic, i.e., (2.1.12) holds. Suppose furthermore that the initial data (4.1.2) satisfy (4.1.3)–(4.1.4). Then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $D(T)$
of the $C^1$ solution $u = u(t, x)$ to the Cauchy problem (4.1.1)–(4.1.2), we have the following uniform a priori estimates:

\[
W_{\infty}^c(T) \leq \kappa_1 \varepsilon, \quad (4.1.5)
\]
\[
\tilde{W}_1(T), \ W_1(T) \leq \kappa_2 \varepsilon, \quad (4.1.6)
\]

and

\[
U_{\infty}(T) \leq \kappa_3 \varepsilon, \quad (4.1.7)
\]

where $\kappa_1, \kappa_2, \text{and} \ \kappa_3$ are positive constants independent of $\varepsilon$ and $T$.

4.2 Formation of Singularities of the $C^1$ Solution in the Noncritical Case $\alpha < +\infty$

In this section we discuss the formation of singularities of the $C^1$ solution to the Cauchy problem (4.1.1)–(4.1.2) in the noncritical case, namely, for the non-WLD hyperbolic system (4.1.1) with finite non-WLD index $\alpha$ (see Section 2.5).

4.2.1 Some Uniform a Priori Estimates Depending on Weak Linear Degeneracy

Lemma 4.2.1 Suppose that in a neighbourhood of $u = 0$, $A(u)$ is suitably smooth and system (4.1.1) is strictly hyperbolic. Suppose furthermore that (4.1.1) is not WLD and its non-WLD index $\alpha$ is finite. Suppose finally that the initial data (4.1.2) satisfy (3.2.1). Then in normalized coordinates there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, on any given existence domain $D(T)$ of the $C^1$ solution $u = u(t, x)$ to the Cauchy problem (4.1.1)–(4.1.2), we have the following uniform a priori estimates:

\[
U_{\infty}^c(T) \leq \kappa_4 \theta \{1 + \theta^{\alpha+2} T\}, \quad (4.2.1)
\]
\[
\tilde{U}_1(T), \ U_1(T) \leq \kappa_5 \theta \{1 + \theta^{\alpha+1} T\}. \quad (4.2.2)
\]

When

\[
T\theta^{\alpha+1} \leq \kappa_6, \quad (4.2.3)
\]

we have

\[
W_{\infty}(T) \leq \kappa_7 \theta, \quad (4.2.4)
\]

in which $U_{\infty}^c(T), \tilde{U}_1(T), \ U_1(T),$ and $W_{\infty}(T)$ are still defined by (3.3.1)–(3.3.4), respectively, and $\kappa_4, \ldots, \kappa_7$ are positive constants independent of $\theta$ and $T$. 