CLIFFORD-ALGEBRAIC RANDOM WALKS
ON THE HYPERCUBE

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Abstract. The \( n \)-dimensional hypercube is a simple graph on \( 2^n \) vertices labeled by binary strings, or \textit{words}, of length \( n \). Pairs of vertices are adjacent if and only if they differ in exactly one position as binary words; i.e., the Hamming distance between the words is one. A discrete-time random walk is easily defined on the hypercube by “flipping” a randomly selected digit from 0 to 1 or vice-versa at each time step. By associating the words as blades in a Clifford algebra of particular signature, combinatorial properties of the geometric product can be used to represent this random walk as a sequence within the algebra. A closed-form formula is revealed which yields probability distributions on the vertices of the hypercube at any time \( k \geq 0 \) by a formal power series expansion of elements in the algebra. Furthermore, by inducing a walk on a larger Clifford algebra, probabilities of self-avoiding walks and expected first hitting times of specific vertices are recovered. Moreover, because the Clifford algebras used in the current work are canonically isomorphic to fermion algebras, everything appearing here can be rewritten using fermion creation/annihilation operators, making the discussion relevant to quantum mechanics and/or quantum computing.

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1. Introduction

The \( n \)-dimensional cube or \textit{hypercube} \( Q_n \) is the simple graph whose vertices are the \( n \)-tuples with entries in \( \{0, 1\} \) and whose edges are the pairs of \( n \)-tuples that differ in exactly one position. This graph has natural applications in computer science, symbolic dynamics, and coding theory [6]. The structure

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of the hypercube allows one to construct a random walk on the hypercube by “flipping” a randomly selected digit from 0 to 1 or vice-versa. Combinatorial properties of Clifford algebras are employed to represent such a random walk as sequences within a particular Clifford algebra. A closed-form formula is revealed which yields probability distributions on the vertices of the hypercube at any time \( k \geq 0 \) from a formal power series expansion of an element in the algebra. By construction of a larger auxiliary Clifford algebra, probabilities of self-avoiding walks and expected first hitting times of specific vertices are recovered.

Let \( b \) be a block, or word, of length \( n \); that is, let \( b \) be a sequence of \( n \) zeros and ones. The weight of \( b \) is defined as the number of 1’s in the sequence. The binary sum of two such words is the sequence resulting from addition modulo-two of the two sequences. The Hamming distance between two binary words is defined as the weight of their binary sum.

**Example 1.1.** Let \( b_1 = 01101100 \) and let \( b_2 = 11001111 \). Then the weights are \( w(b_1) = 4 \) and \( w(b_2) = 6 \). The Hamming distance between \( b_1 \) and \( b_2 \) is \( w(10100011) = 4 \).

A simple graph \( G = (V, E) \) is a set \( V \) of vertices and a set \( E \) of unordered pairs of vertices, or edges, such that no unordered pair occurs more than once in \( E \), and there is no unordered pair of the form \( \{v, v\} \). Another way of saying this is that a simple graph contains no multiple edges and no loops. Two vertices \( v_1, v_2 \in V \) are said to be adjacent if \( \{v_1, v_2\} \in E \).

**Definition 1.2.** The \( n \)-dimensional hypercube \( Q_n \) is the simple graph whose vertices are the \( 2^n \) \( n \)-tuples from \( \{0, 1\} \) and whose edges are defined by the rule

\[
\{v_1, v_2\} \in E(Q) \text{ iff } w(v_1 + v_2) = 1.
\]

Here \( v_1 + v_2 \) is bitwise addition modulo-two, and \( w \) is the weight. In other words, two vertices of the hypercube are adjacent if and only if their Hamming distance is 1.

What follows in the remainder of this introductory subsection is a more-or-less standard introduction to Clifford algebras and their properties. The reader is referred to works such as [3] and [4] for essential background information on Clifford algebras.