An Introduction to Commutative Quaternions

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Abstract. A Scheffers theorem states that for commutative hypercomplex numbers the differential calculus does exist and the functions can be introduced in the same way as they are for the complex variable. This property could open new applications of commutative quaternions in comparison with non-commutative Hamilton quaternions.

In this article we introduce some quaternionic systems, their algebraic properties and the differential conditions (Generalized Cauchy-Riemann conditions) that their functions must satisfy.

Then we show that the functional mapping, studied in the geometry associated with the quaternions, does have the same properties of the conformal mapping performed by the functions of complex variable. We also summarize the expressions of the elementary functions.

1. Introduction

In this article we apply the geometrical considerations exposed in [1] to some of the four-dimensional hypercomplex numbers that in [1] have been called “generalized Segre quaternions” [2].

Some years after their geometrical introduction these numbers were reconsidered by F. Severi [3] in association with the functions of two complex variables and, after him by G. Scorza-Dragoni [4] and U. Morin [5], who studied the existence and the differential properties of their functions. Here we remember that, from a physical point of view, there is an important difference between the functions of two complex variables and the functions of commutative quaternions. In fact while for the former two real functions of four real variables are defined, for the latter four real functions of four real variables are introduced.

We know that while the complex numbers can represent vectors, their functions represent vector fields; the same applications can hold for the quaternions and for their functions. We shall see that the representation by means of hypercomplex numbers gives automatically both to vectors and vector fields the same symmetries (groups, geometries) and then the same invariants, since the structure constants
act as “symmetry preserving operator” between vectors and the corresponding vectors fields. These symmetries are the ones of the geometries “generated” by hypercomplex numbers [1].

Vectors and vector fields are usually represented in Euclidean geometry, but whereas for complex numbers the Euclidean geometry is their own geometry the same is not true for all the other hypercomplex numbers.

We shall see that the commutative quaternions, when are represented in their geometry, keep the same algebraic and differential properties of the complex numbers. In particular the conformal mappings have the same properties of the functions of complex variable.

If a physical meaning may be associated with these “geometries”, the problem we consider can be simplified, as well as it happens if we use the polar coordinates for problems with spherical symmetry or the hyperbolic numbers for the space-time symmetry [6].

Let us come back to Segre quaternions: in recent time G. B. Price [7] has developed, with the necessary mathematical rigour, the results obtained by [5, 8] and C. Davenport [23] has shown the importance of the differential calculus associated with hypercomplex numbers. In our expositions we start from the work [5] and apply the considerations about the invariant and geometries exposed in [1]. Also for the style we follow [5] and refer to [7] for rigorous mathematical demonstrations.

The paper is organized in the following way: in sect. 2 the algebraic properties of three systems of Segre quaternions are summarized. In sect. 3 the functions of quaternionic variables are introduced and the Generalized Cauchy-Riemann conditions are obtained. In sect. 4 the properties of the conformal mapping are demonstrated. In the appendices we give the elementary functions of the quaternions and introduce another system of commutative quaternions.

2. Commutative Quaternions

2.1. Algebraic properties of commutative quaternions

In this paper we consider, among all the hypercomplex number systems $H_n$ [9], those ones with $n = 4$. Among these ones we consider some of the systems that have been called generalized Segre quaternions [1]; in particular the quaternions that, as we shall see, are decomposable into two bidimensional hypercomplex systems that we indicate by $H_2$. These bidimensional systems are the systems that have been considered by [10, 11, 12]. We also indicate by $Q$ the set of quaternions, that we define:

$$\{q = t + ix + jy + kz; \ t, x, y, z \in \mathbb{R}; \ i, j, k \notin \mathbb{R}\},$$

(1)