Clifford Fourier Transformation and Uncertainty Principle for the Clifford Geometric Algebra $Cl_{3,0}$

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Abstract. First, the basic concept of the vector derivative in geometric algebra is introduced. Second, beginning with the Fourier transform on a scalar function we generalize to a real Fourier transform on Clifford multivector-valued functions ($f : \mathbb{R}^3 \to Cl_{3,0}$). Third, we show a set of important properties of the Clifford Fourier transform on $Cl_{3,0}$ such as differentiation properties, and the Plancherel theorem. Finally, we apply the Clifford Fourier transform properties for proving an uncertainty principle for $Cl_{3,0}$ multivector functions.

Keywords. vector derivative, multivector-valued function, Clifford (geometric) algebra, Clifford Fourier transform, uncertainty principle.

1. Introduction

In the field of applied mathematics the Fourier transform has developed into an important tool. It is a powerful method for solving partial differential equations. The Fourier transform provides also a technique for signal analysis where the signal from the original domain is transformed to the spectral or frequency domain. In the frequency domain many characteristics of the signal are revealed. With these facts in mind, we extend the Fourier transform in geometric algebra.

Brackx et al. [11] extended the Fourier transform to multivector valued function-distributions in $Cl_{0,n}$ with compact support. They also showed some properties of this generalized Fourier transform. A related applied approach for hypercomplex Clifford Fourier Transformations in $Cl_{0,n}$ was followed by Bülow et. al. [7]. In [13], Li et. al. extended the Fourier Transform holomorphically to a function of $m$ complex variables.
In this paper we adopt and expand the generalization of the Fourier transform in Clifford geometric algebra\(^1\) \(G_3\) recently suggested by Ebling and Scheuermann [3]. We explicitly show detailed properties of the real\(^2\) Clifford geometric algebra Fourier transform (CFT), which we subsequently use to define and prove the uncertainty principle for \(G_3\) multivector functions.

We start with a review of the vector derivative for a multivector valued function. We demonstrate that with a little modification it obeys rules which resemble the rules for a scalar partial derivative.

2. Clifford’s Geometric Algebra

In this section we introduce the axioms and the vector derivative of geometric algebra. Fore more details we refer the reader to [2, 5].

2.1. Axioms of geometric algebra

For \(G_n\) to be a Clifford geometric algebra over the real \(n\)-dimensional Euclidean vector space \(\mathbb{R}^n\), the geometric product of elements \(A, B, C \in G_n\) must satisfy the following axioms:

**Axiom 1.** Addition is commutative:

\[ A + B = B + A. \]

**Axiom 2.** Addition and the geometric product are associative:

\[ (A + B) + C = A + (B + C), \quad A(BC) = (AB)C, \]

and distributive:

\[ A(B + C) = AB + AC, \quad (A + B)C = AC + BC. \]

**Axiom 3.** There exist unique additive and multiplicative identities \(0\) and \(1\) such that:

\[ A + 0 = A, \quad 1A = A. \]

**Axiom 4.** Every \(A\) in \(G_n\) has an additive inverse:

\[ A + (-A) = 0. \]

**Axiom 5.** For any nonzero vector \(a\) in \(G_n\) the square of \(a\) is equal to a unique positive scalar \(|a|^2\), that is

\[ aa = a^2 = |a|^2 > 0. \]

\(^1\)In the geometric algebra literature [2] instead of the mathematical notation \(Cl_{p,q}\) the notation \(G_{p,q}\) is widely in use. It is convention to abbreviate \(G_{n,0}\) to \(G_n\).

\(^2\)The meaning of real in this context is, that we use the three dimensional volume element \(i_3 = e_{123}\) of the geometric algebra \(G_3\) over the field of the reals \(\mathbb{R}\) to construct the kernel of the Clifford Fourier transformation of definition 3. This \(i_3\) has a clear geometric interpretation.