Representation of Probabilistic Data by Quantum-Like Hyperbolic Amplitudes

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Abstract. In this paper we study the problem of representation of statistical data of any origin by hyperbolic probabilistic amplitudes (normalized vectors in hyperbolic Hilbert space). It generalizes the conventional QM which is based on complex Hilbert space. We performed extended numerical simulation. Similar to the conventional quantum formalism for Bloch’s sphere, we visualize results of simulation for a special class of statistical data on so called Bloch’s hyperboloid. The notion of hyperbolic qubit is introduced.

Keywords. Representation of probabilistic data by hyperbolic amplitudes, Bloch’s sphere, Bloch’s hyperboloid.

1. Introduction

Applications of Clifford algebras in physics is well established domain of modern research, see e.g. [1] for detailed review. In particular, applications of so called hyperbolic numbers to quantum theory were considered by various authors. Hyperbolic quantization appeared naturally in relativistic quantum physics. The hyperbolic numbers offer the possibility to represent the four-component Dirac spinor as a two component hyperbolic spinor. Hucks has shown [2] that the Lorentz group is equivalent to the hyperbolic unitary group. Porteous [3] proved the unitarity of special linear group with the help of the double field, which corresponds to the null basis representation of the hyperbolic numbers. Ulrych investigated the hyperbolic representation of Poincare mass [4]–[6]. He also studied symmetries in the hyperbolic Hilbert space. Applications of hyperbolic numbers in general relativity can be found in the paper [7] of Kunstatter et al.

We emphasize that QM is based on two coupled mathematical structures: algebraic and probabilistic ones. The algebraic structure of the conventional QM is given by complex Hilbert space. Its probabilistic structure is coupled to the algebraic one via Born’s rule.
Consider a (pure) quantum state $\psi$. It is represented by a normalized vector of complex Hilbert space. Consider also observable $a$. It is represented by a self-adjoint operator $\hat{a}$. Assume (for simplicity) that it has purely discrete non-degenerate spectrum. By Born’s rule the probability to obtain some value $\alpha$ of observable $a$ is given by the square of the absolute value of projection of $\psi$ onto the eigenvector of the operator $\hat{a}$ corresponding to eigenvalue $\alpha$.

There are two possibilities to generalize the conventional quantum formalism: either to generalize directly its algebraic structure and to produce a new calculus of probabilities or to start with the problem of creation of a vector representation of probabilistic data and to induce new algebraic structures. I tried to proceed in both ways, see [8] for the first approach: from the hyperbolic Hilbert space model to a new type of interference of probabilities, and see [9] for the second approach: from generalizations of the formula of total probability to the hyperbolic Hilbert space model.

1.1. Formula of total probability and its deformations

We recall the conventional formula of total probability:

$$P(b = \beta_j) = P(a = \alpha_1)P(b = \beta_j|a = \alpha_1) + P(a = \alpha_2)P(b = \beta_j|a = \alpha_2).$$

Here $b = \beta_1, \beta_2$ and $a = \alpha_1, \alpha_2$ are two dichotomous random variables; $P(b = \beta_j)$ is the probability for finding the observable $b$ to have the value $\beta_j$ under a context in that the observable $a$ can take either of two values, namely, $\alpha_1$ and $\alpha_2$, with respective probabilities $P(a = \alpha_1)$ and $P(a = \alpha_2)$. Thus the probability $P(b = \beta)$ can be reconstructed on the basis of transition probabilities $P(b = \beta|a = \alpha)$.

This formula plays the fundamental role in classical statistics and decision making. However, it is violated in experiments with quantum systems.

In a series of papers on probabilistic foundations of quantum theory, see e.g. [9], we demonstrated that depending on observables the conventional formula of total probability can be deformed in two ways. We did not start with any algebraic structure. We were interested in deformations of the formula of total probability.

1.2. Trigonometric interference

One deformation of the formula of total probability is the well known formula for quantum interference of probabilities:

$$P(b = \beta_j) = P(a = \alpha_1)P(b = \beta_j|a = \alpha_1) + P(a = \alpha_2)P(b = \beta_j|a = \alpha_2) + 2 \cos \phi_j \sqrt{P(b = \beta_j|a = \alpha_1)P(a = \alpha_1)P(b = \beta_j|a = \alpha_2)P(a = \alpha_2)}.$$  

(1)

The conventional formula of total probability is perturbed by the trigonometric term. Such a “formula of total probability with interference term” can be easily

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1 By the spectral postulate of QM it belongs to spectrum of $\hat{a}$.

2 The prior probability to obtain the result e.g. $b = \beta$ is equal to the prior expected value of the posterior probability of $b = \beta$ under conditions $a = \alpha$. 