Variational Formulation for Quaternionic Quantum Mechanics

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Abstract. A quaternionic version of Quantum Mechanics is constructed using the Schwinger’s formulation based on measurements and a Variational Principle. Commutation relations and evolution equations are provided, and the results are compared with other formulations.

Keywords. Quaternionic quantum mechanics, variational principle.

1. Introduction

In 1936 Birkhoff and von Neumann [1] have shown the existence of a propositional calculus as fundamental ingredient of Quantum Mechanics (QM), which could be written using only the outputs of measures. It does not assume any set of numbers or even a particular vectorial space, but contains the essentials of QM such as uncertainty relations and complementary properties. Of course, the authors showed that there are three different realizations for this propositional calculus, corresponding to the real or complex numbers or still quaternions. Octonions and higher dimensional extensions of the complex numbers are discarded, since they cannot have a conservation law for the probability current [2].

We can ask: which of these three realizations of the “general” QM of Birkhoff and von Neumann is present in Nature? Here it is implicit the hypothesis that the set of numbers of a given theory reflects part of the physical information about the system. While the differences between the real and complex QM are relatively simple and well known [3], the quaternionic version has many new and rich characteristics. Therefore, it sounds strange that such possibility is not much explored, but there are very good reasons for this. First, the problem of writing a quaternionic Schrödinger equation is not trivial since it involves the explicit use of imaginary unit. Second, the representation of composite systems by a direct
The classical theory of physical measurements is based on the concept that the interaction between the system under observation and the measurement apparatus can be done arbitrarily small or, at least, precisely compensated, in such way to specify an idealized measurement which does not disturb any other property of the system. However, the experiment had demonstrated that the interaction can not be done arbitrarily small neither the disturb produced can be precisely compensated since it is uncontrollable and unpredictable. The fact that the interaction can not be arbitrarily small is expressed by the finite size of the Planck constant, while the uncontrollable character of the interaction is given by the uncertainty principle. Therefore, the measurement of a given property can produce a significant change in the value of another previously measured property, and then there is no sense in speaking about an microscopic system with definite values for all its attributes. This is in contradiction with the classical representation of physical quantities by numbers. The laws of a microscopic physical system must then be expressed in a non-classical mathematical language constituting a symbolic expression of the properties of microscopic measurements.

In what follows, we will develop the general lines of such mathematical structure discussing about simplified physical systems where any physical quantity $A$ can have only a finite number of different values $a_1, a_2, a_3, \ldots$. The most simple measurement consider an ensemble of similar independent systems which is divided by the apparatus of measurement in sub-ensembles distinguished by the defined values of the physical quantity under measurement. Let us denote $M_a$ the selective measurement accepting any system having value $a$ for the property $A$ and rejecting any other. The addition of such symbols is defined as implying a less specific measure, resulting in a sub-ensemble associated with any value under the sum, none of them being distinguished of the others by the measurement.

The multiplication of measurement symbols implies the sequence of measurements reading from right to left. From the physical meaning of such operations, we learn that addition is commutative and associative while multiplication is only associative. Using $\hat{1}$ and $\hat{0}$ to represent respectively the measures which accept and reject all systems, the properties of the elementary selective measurement are...