Cauchy-Like Integral Formula for Functions of a Hyperbolic Variable

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Abstract. It has been recently shown that, working in a plane with the geometry associated with hyperbolic numbers, a complete “Euclidean” formalization of geometry and trigonometry of Minkowski space-time has been obtained.

In this paper we show a first result that indicates that also for the studies of functions of a hyperbolic variable, the more appropriate way is to work in a plane with pseudo-Euclidean metric.

In particular it is shown that a theorem equivalent to the Cauchy integral formula for the functions of a complex variable holds for the functions of a hyperbolic variable studied in the pseudo-Euclidean plane.

Keywords. Hyperbolic numbers, functions of a hyperbolic variable, wave equation.

1. Introduction

“The extraordinary apt way in which the properties of functions of a complex variable fit our need for solutions of two-dimensional Laplace equation, is completely lost for other equations and more dimensions” [1, p. 1252].

This difficulty also holds for the wave equation, notwithstanding its similarity with Laplace equation (a partial differential equation of second order with constant coefficients).

In recent times a two-dimensional system of number, called hyperbolic, has been widely studied and the comparison with complex numbers has allowed an exhaustive formalization of space time geometry and trigonometry [2], [3, 4]. Otherwise for hyperbolic numbers too a theory of functions of a hyperbolic variable has been formalized [5, Chap. 7].

These considerations stimulate us to begin a research field about the possibilities of using the functions of a hyperbolic variable for studying the solutions of wave equation that represents for these numbers what Laplace equation does for complex numbers.
In this paper we begin a study by using an approach deriving from the following associations [5]:

Euclidean Geometry $\rightarrow$ complex numbers $\rightarrow$ functions of a complex variable $\rightarrow$ Laplace equation $\Rightarrow$ wave equation $\rightarrow$ functions of a hyperbolic variable $\rightarrow$ hyperbolic numbers $\rightarrow$ space-time geometry $\rightarrow$ study in Minkowski geometry.

In this paper we show a first result: to state that a \textit{Cauchy-like integral formula} holds for functions of a hyperbolic variable if they are studied in a plane with non-definite metric.

2. Properties of Functions of a Hyperbolic Variable

For the functions of a complex variable, if a function is known on the frontier of a connected domain and suitable regularity conditions [8], [9] are satisfied, the Cauchy integral theorem allows one to calculate the function in all the internal points. From a topological point of view this domain is equivalent to a circle.

In the hyperbolic plane the curve corresponding to the Euclidean circle is the equilateral hyperbola [2].

The equilateral hyperbola of a given kind also agrees with the Special Relativity requirement that a curve in the space-time plane, represent a possible motion if its tangent lines are of the same kind (time-like). More the equilateral hyperbolas for many other properties [2], are the equivalent of circles in Euclidean plane. For this last property, in the present paper, the equilateral hyperbolas are the natural substitute of the circles used in the studies of functions of a complex variable.

Now we see how the “simple nature” of hyperbolic numbers and, in particular, the decomposability property of hyperbolic numbers and of functions of a hyperbolic variable [5, Sect. 2.2.2] (also recalled in App. A.3), allows us to state the following

\textbf{Theorem.} \textit{If the values of a function of a hyperbolic variable are known, on an arm of equilateral hyperbola (as, for instance, $\in R_s$ (right sector) [2]) the values of the function in the sector can be calculated.}

\textit{Proof.} Let us refer to Fig. 1, in which the hyperbolic variable $z = x + hy$ is considered in the right sector ($R_s$) of the hyperbolic plane ($x, y$). Let us consider a point $P \equiv (x, y) \equiv (z)$. internal to the hyperbola\(^1\) and the two straight lines parallel to axes bisectors (characteristic lines) through the point $P$ and let us call $P_1$ and $P_2$ the points in which the characteristic lines intersect the hyperbola.

The functions of the hyperbolic variable may be expressed in a decomposed form (see App. A.1) and represented in the plane ($\xi = x+y$, $\eta = x-y$) of Fig. 2.

Let us write the function of a hyperbolic variable as

\[ f(z) = U + hV, \]  

\(^1\)Following [2] are defined as internal the points between the hyperbola and its asymptote.