Induced Representations and Hypercomplex Numbers

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Abstract. In the search for hypercomplex analytic functions on the half-plane, we review the construction of induced representations of the group $G = \text{SL}_2(\mathbb{R})$. Firstly we note that $G$-action on the homogeneous space $G/H$, where $H$ is any one-dimensional subgroup of $\text{SL}_2(\mathbb{R})$, is a linear-fractional transformation on hypercomplex numbers. Thus, we investigate various hypercomplex characters of subgroups $H$. The correspondence between the structure of the group $\text{SL}_2(\mathbb{R})$ and hypercomplex numbers can be illustrated in many other situations as well. We give examples of induced representations of $\text{SL}_2(\mathbb{R})$ on spaces of hypercomplex valued functions, which are unitary in some sense. Raising/lowering operators for various subgroup prompt hypercomplex coefficients as well.

Keywords. Induced representation, unitary representations, $\text{SL}_2(\mathbb{R})$, semisimple Lie group, complex numbers, dual numbers, double numbers, Möbius transformations, split-complex numbers, parabolic numbers, hyperbolic numbers, raising/lowering operators, creation/annihilation operators.

1. Introduction

Analytic functions of a complex variable form a beautiful theory with rich applications in many fields ranging from number theory to electrical engineering. Thus, it is natural to look for its analogs and generalisations in different directions. The most basic (or fundamental?) situation appears if we replace the complex imaginary unit $i^2 = -1$ with either the hyperbolic one $j^2 = +1$ or the nilpotent $\varepsilon^2 = 0$.

Two-dimensional commutative associative algebra over reals generated by 1 and $j$ consists of numbers $x + jy$, where $x, y \in \mathbb{R}$. They are known as split-complex, duplex, hyperbolic or double numbers [4, 14, 43, 50]. The algebra has zero divisors $j_\pm = \frac{1}{2}(1 \pm j)$ with the properties $j_\pm^2 = j_\pm$ and $j_+j_- = 0$. Thus, double numbers are isomorphic to $\mathbb{R} \oplus \mathbb{R}$—the direct sum.

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of two copies of the real line spanned by $j_+$ and $j_-$. This explains the names “split-complex” and “double”.

The analogous algebra associated to the nilpotent unit $\varepsilon$ consists of elements $x + \varepsilon y$, which are called dual numbers [6, 12, 54]. All zero divisors in the algebra are $\varepsilon y$, $y \in \mathbb{R}$. Physical applications of hypercomplex numbers are scattered through classical mechanics [31, 54], non-linear dynamics [43, 44], relativity [4, 50], cosmology [9, 12] and quantum mechanics [14, 34, 51].

**Remark 1.1.** Unfortunately, there are no common notations for hypercomplex units. Moreover, it would be difficult simply to list the whole variety of symbols employed for this. Even the complex imaginary unit $i$ is oftenly written as $j$ in engineering. The hyperbolic unit is denoted by $j$ in many papers starting at least from the foundational article [53]; although a different letter $e$ is used in the remarkable book [54]. The symbol $i$ is used for the nilpotent unit in [10, 45], however we chose $\varepsilon$ following Yaglom [54]. The latter notation is also suggestive in light of the following remark.

**Remark 1.2.** The parabolic unit $\varepsilon$ is a close relative to the infinitesimal number $\varepsilon$ from non-standard analysis [8, 52]. The former has the property that its square is exactly zero, meanwhile the square of the latter is almost zero at its own scale. In fact, there is a version of non-standard analysis [3] employing the nilpotent unit $\varepsilon$ as an infinitesimal\(^1\). Also, some non-standard proofs of the main calculus theorems are given in [6]. A similar property allows to obtain classical mechanics from the representations of the Heisenberg group [31, 34].

*What kind of “analytic” functions can be associated with dual and double numbers?* Since this question is very natural it was addressed over a prolonged period of time by various researchers. Many of them were unaware of works of their predecessors, neither I can claim to possess the complete knowledge. Below is a brief summary of several works known to me.

For double numbers, a systematic study was already accomplished in [53], there are also numerous later investigations and surveys, see [4, 14], [37, Part IV], [39, 42, 43, 48] and further references therein. The existing consensus is based on the factorisation of the wave equation $\partial_x^2 - \partial_y^2 = (\partial_x - j \partial_y)(\partial_x + j \partial_y)$ into a product of two linear differential operators. This is an analog of the factorisation of the Laplacian $\partial_x^2 + \partial_y^2 = (\partial_x - i \partial_y)(\partial_x + i \partial_y)$ into the product of the Cauchy–Riemann operator and its adjoint. Thus, hyperbolic analytic functions are defined to be null solutions of the operator $\partial_x + j \partial_y$. However, the split of dual number in the basis $j_\pm$ reduces an “analytic” function $f(x, y) = j_+ f_+(x + y) - j_- f_-(x - y)$ to the sum, where $f_\pm$ are two generic differentiable real-valued functions of a real variable. This is related to the representation of the generic solution of the wave equation on the infinite string as a sum of a wave traveling to the left and another traveling to the right.

\(^1\)I am grateful to the anonymous referee for pointing my attention to the book [3] by J.L. Bell.