Cyclic Structures of Cliffordian Supergroups and Particle Representations of $\text{Spin}^+_+(1,3)$

V. V. Varlamov

Abstract. Supergroups are defined in the framework of $\mathbb{Z}_2$-graded Clifford algebras over the fields of real and complex numbers, respectively. It is shown that cyclic structures of complex and real supergroups are defined by Brauer-Wall groups related with the modulo 2 and modulo 8 periodicities of the complex and real Clifford algebras. Particle (fermionic and bosonic) representations of a universal covering (spinor group $\text{Spin}^+_+(1,3)$) of the proper orthochronous Lorentz group are constructed via the Clifford algebra formalism. Complex and real supergroups are defined on the representation system of $\text{Spin}^+_+(1,3)$. It is shown that a cyclic (modulo 2) structure of the complex supergroup is equivalent to a supersymmetric action, that is, it converts fermionic representations into bosonic representations and vice versa. The cyclic action of the real supergroup leads to a much more high-graded symmetry related with the modulo 8 periodicity of the real Clifford algebras. This symmetry acts on the system of real representations of $\text{Spin}^+_+(1,3)$.

Keywords. Clifford algebras, supergroups, spinor representations.

1. Introduction

It is well known that Clifford algebras (systems of hypercomplex numbers) have a broad application in many areas of mathematical and theoretical physics. From historical point of view, Clifford algebras have essentially geometric origin [1], because they are the synthesis of Hamilton quaternion calculus [2] and Grassmann Ausdehnungslehre [3], and by this reason they called by Clifford as geometric algebras [4]. Further, Lipschitz [5] showed that Clifford algebras are related closely with the study of rotation groups of multidimensional spaces. After fundamental works of Cartan [6], Witt [7] and Chevalley [8] the Clifford algebra theory takes its modern form [9, 10, 11]. It is known that Clifford algebras inherit $\mathbb{Z}_2$-graded structure from the Grassmann algebras. This fact allows one to define Cliffordian supergroups using a classical theory of formal Lie groups [12, 13]. Along with the $\mathbb{Z}_2$-graded
structure Clifford algebras possess modulo 2 and modulo 8 periodicities over the fields of complex and real numbers, respectively (see [14, 15, 16]). These periodic relations are described by Brauer-Wall groups [17, 18]. The cyclic (modulo 2 and 8) structure is the most essential property of the Cliffordian supergroups.

In the present work we consider the field \( \psi(\alpha) = \langle x, g | \psi \rangle \) on the representation spaces of a spinor group \( \text{Spin}_+(1,3) \), where \( x \in T_4 \) and \( g \in \text{Spin}_+(1,3) \) (\( T_4 \) is a translation subgroup of the Poincaré group \( \mathcal{P} \)). At this point, four parameters \( x^\mu \) correspond to position of the point-like object, whereas remaining six parameters \( g \in \text{Spin}_+(1,3) \) define orientation in quantum description of orientable (extended) object [19, 20]. In general, the field \( \psi(\alpha) \) is defined on the homogeneous space \( M_{10} = \mathbb{R}_{1,3} \times \mathfrak{L}_6 \) (a group manifold of \( \mathcal{P} \)), where \( \mathbb{R}_{1,3} \) is the Minkowski spacetime and \( \mathfrak{L}_6 \) is a group manifold of \( SO_0(1,3) \). On the other hand, the space \( M_{10} = \mathbb{R}_{1,3} \times \mathfrak{L}_6 \) can be understood as a fiber bundle, where a bundle base is \( \mathbb{R}_{1,3} \). All intrinsic properties of the field \( \psi(\alpha) \) are described within the group \( \text{Spin}_+(1,3) \) and its representations.

The paper is organized as follows. A short introduction to the Clifford algebra theory is given in section 2. \( \mathbb{Z}_2 \)-graded Clifford algebras, Brauer-Wall supergroups and Trautman diagrams (spinorial clocks) are considered in section 3 over the fields of real and complex numbers. Complex and real representations of the field \( \psi(\alpha) = \langle x, g | \psi \rangle \) of different types are constructed in section 4 within representations of \( \text{Spin}_+(1,3) \). A relationship between tensor products of the Clifford algebras and a Gel’fand-Naimark representation basis of the proper orthochronous Lorentz group is established. Complex and real supergroups are defined on the representation systems of \( \text{Spin}_+(1,3) \) in section 5.

2. Algebraic Preliminaries

In this section we will consider some basic facts concerning Clifford algebras. Let \( F \) be a field of characteristic 0 (\( F = \mathbb{R}, F = \mathbb{C} \)), where \( \mathbb{R} \) and \( \mathbb{C} \) are the fields of real and complex numbers, respectively. A Clifford algebra \( \mathcal{C} \) over a field \( F \) is an algebra with \( 2^n \) basis elements: \( e_0 \) (unit of the algebra) \( e_1, e_2, \ldots, e_n \) and products of the one-index elements \( e_{i_1i_2\ldots i_k} = e_{i_1}e_{i_2}\ldots e_{i_k} \).

Over the field \( F = \mathbb{R} \) the Clifford algebra is denoted as \( \mathcal{C}_{p,q} \), where the indices \( p \) and \( q \) correspond to the indices of the quadratic form

\[
Q = x_1^2 + \ldots + x_p^2 - \ldots - x_{p+q}^2
\]

of a vector space \( V \) associated with \( \mathcal{C}_{p,q} \). The multiplication law of \( \mathcal{C}_{p,q} \) is defined by a following rule:

\[
e_i^2 = \sigma(p - i)e_0, \quad e_ie_j = -e_je_i,
\]

where

\[
\sigma(n) = \begin{cases} 
-1 & \text{if } n \leq 0, \\
+1 & \text{if } n > 0.
\end{cases}
\]