Dual Fibonacci Quaternions

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Abstract. In this study, we define the dual Fibonacci quaternion and the dual Lucas quaternion. We derive the relations between the dual Fibonacci and the dual Lucas quaternion which connected the Fibonacci and the Lucas numbers. Furthermore, we give the Binet and Cassini formulas for these quaternions.

Keywords. Fibonacci quaternion, dual Fibonacci quaternion, dual Lucas quaternion.

1. Introduction

The algebra of quaternions is denoted by \( \mathbb{H} \), also by the Clifford algebra classifications \( Cl_{0,2}(R) \cong Cl_{3,0}(R) \). A quaternion is defined in the form

\[
q = q_0 + iq_1 + jq_2 + kq_3
\]

where \( q_0, q_1, q_2, q_3 \) are real numbers and \( i, j, k \) are standard orthonormal basis in \( \mathbb{R}^3 \) which satisfy the quaternion multiplication rules as

\[
\begin{align*}
i^2 & = j^2 = k^2 = -1 \\
i j & = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.
\end{align*}
\]

Since we will clarify the dual Fibonacci quaternions, we now give the basic notations of dual quaternions.

Clifford [2] published his work on dual numbers in 1873 and provided us with a powerful tool for the analysis of complex numbers. The dual numbers extend to the real numbers has the form

\[d = a + \varepsilon a^*\]

where \( \varepsilon \) is the dual unit and \( \varepsilon^2 = 0, \varepsilon \neq 0 \).

A dual quaternion is an extension of dual numbers whereby the elements of that quaternion are dual numbers. Dual quaternions are used as an appliance for expressing and analyzing the physical properties of rigid bodies.

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They are computationally efficient approach of representing rigid transforms like translation and rotation.

The dual quaternion is represented in the form

$$Q = q + \varepsilon q^*$$

where \(q\) and \(q^*\) are quaternions and \(\varepsilon\) is the dual unit.

If \(q = q_0 + iq_1 + jq_2 + kq_3\) and \(q^* = q_0^* + iq_1^* + jq_2^* + kq_3^*\), then the dual quaternion \(Q\) can be denoted as:

$$Q = (q_0 + \varepsilon q_0^*) + i(q_1 + \varepsilon q_1^*) + j(q_2 + \varepsilon q_2^*) + k(q_3 + \varepsilon q_3^*).$$

So the dual quaternion \(Q\) is constructed from eight real parameters.

From these notations it can be said that the above properties are dual version of quaternions.

There are many works on Fibonacci and Lucas numbers. Dunlap [3], Vajda [14], Verner [15] and Hoggatt [15] explained the properties of Fibonacci and Lucas numbers and computed the relations between them.

The \(n^{th}\) Fibonacci and \(n^{th}\) Lucas quaternions were described by Horadam in [8] as

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} \quad (1.2)$$

and

$$K_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3} \quad (1.3)$$

respectively, where as throughout this paper, \(F_n\) is the \(n^{th}\) Fibonacci number with the initial values \(F_0 = 1, F_1 = 1\) and \(L_n\) is the \(n^{th}\) Lucas number with the initial values \(L_0 = 2, L_1 = 1\). Also \(i, j, k\) are standard orthonormal basis in \(\mathbb{R}^3\).


In [4], Halıcı expressed the generating function and Binet formulas for these quaternions. Akyiğit, Kösal and Tosun [1] defined the split Fibonacci and split Lucas quaternions. They also gave Binet formulas and Cassini identities for these quaternions.

In this paper, we define the dual Fibonacci quaternion and the dual Lucas quaternion by combining Fibonacci, Lucas quaternions and dual quaternions. We find the equations between the given quaternions and give the Binet and Cassini formulas for them.

2. Dual Fibonacci Quaternions

Complex Fibonacci numbers are given in [8] by Horadam as;

$$C_n = F_n + iF_{n+1}, \quad i^2 = -1.$$  

Also Halıcı [5] described the \(n^{th}\) complex Fibonacci quaternion as follows;

$$R_n = Q_n + iQ_{n+1}, \quad i^2 = -1$$