One-Parameter Planar Motions in Generalized Complex Number Plane $\mathbb{C}_J$

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Abstract. In this study, we firstly give the basic notations of the generalized complex number plane ($p$-complex plane) $\mathbb{C}_p$. Then, we introduce the one-parameter planar motions in $p$-complex plane $\mathbb{C}_J$ such that $\mathbb{C}_J \subset \mathbb{C}_p$. These motions correspond the one-parameter motions in affine Cayley-Klein planes [1]. We examine this motion theory with aspects of complex motions as given in [2]. Besides, we discuss the relations between absolute, relative, sliding velocities (accelerations) and pole curves under the motions $\mathbb{C}_J/\mathbb{C}_J'$.

Keywords. Generalized complex number plane, complex-type numbers, one-parameter planar motion, kinematics.

1. Introduction

Alternative definitions of the imaginary unit $i$ other than $i^2 = -1$ can give rise to interesting and useful complex number systems. The Italian mathematicians G. Gardan (1501-1576) and R. Bombelli (1526-1572) are thought to be the first people to utilize the complex numbers with taking imaginary unit $i$ such as $i^2 = -1$. After that, researchers have modified the original definition of product of complex numbers. The English geometer W. Clifford (1845-1879) developed the double complex numbers (perplex numbers [3], split-complex numbers [4] or hyperbolic numbers [4]-[5], [6]-[10]) by requiring that $i^2 = 1$. Clifford’s application of double numbers to mechanics has been supplemented by applications to non-Euclidean geometries. The German geometer E. Study (1862-1930) added another variant to the collection of complex products. The dual numbers provide the condition that $i^2 = 0$ [5]. The use of dual number methods for the analysis of spatial mechanisms, robotic control and virtual reality has a significant role in kinematics [11]. With a further aspect, the dual number application to rigid body kinematics was generalized by the principle of transference [11]-[13]. This principle

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points out that when dual numbers replace real ones, then all relations of vector algebra for intersecting lines are also valid for skew lines. The groundwork of spatial instantaneous kinematic differential geometry is studied and the set of lines with special kinematic meaning in moving space is examined by using dual vector calculus in [14]. Moreover, the algebra of dual quantities, the differential geometry of dual curves, the facts about dual curves and dual motions and their application to theoretical space kinematics are discussed in [15].

The ordinary, dual and double numbers are significant members of a two-parameter family of complex number systems often called binary numbers or generalized complex numbers [16]. In the literature; ordinary, dual, and double numbers are usually denoted by different imaginary units, $i, \varepsilon$ and $j$, respectively. For generalizing this unit, we will take $J$ for three number systems. So, the generalized complex numbers have the form [17]

$$z = x + Jy, \quad (x, y \in \mathbb{R}) \quad \text{where} \quad J^2 = i q + p, \quad (p, q \in \mathbb{R}).$$

The generalized complex number systems are isomorphic (as rings) to the ordinary, dual and double numbers when $p + q^2/4$ is negative, zero, and positive, respectively [5].

With taking $J^2 = p; q = 0$ and $-\infty < p < \infty$, generalized complex number system can be presented as follows:

$$\mathbb{C}_p = \{x + Jy : x, y \in \mathbb{R}, \ J^2 = p\}.$$

$\mathbb{C}_p$ is called $p$-complex plane [16]. Moreover, the set $\mathbb{C}_J$ is defined

$$\mathbb{C}_J = \{x + Jy : x, y \in \mathbb{R}, \ J^2 = p, \ p \in \{-1, 0, 1\}\} \quad (1.1)$$

such that $\mathbb{C}_J \subset \mathbb{C}_p$.

The set $\mathbb{C}_J$ is just the real numbers extended to include the unipotent $J$ such that $\mathbb{C}_J := \mathbb{P}_e[J]$, where $\mathbb{P}_e$ represents affine Cayley-Klein planes [18]. This yields by the same extension of the set of ordinary (complex) numbers $\mathbb{C}$, dual numbers $\mathbb{D}$ and double (hyperbolic) numbers $\mathbb{H}$ such that $\mathbb{C} := \mathbb{R}[i], \mathbb{D} := \mathbb{R}[\varepsilon]$ and $\mathbb{H} := \mathbb{R}[j]$, respectively [19]-[20]. The $p$-complex numbers system play the same role for Cayley-Klein geometry like that played by ordinary numbers in the Euclidean geometry [5], [18]. The Cayley-Klein plane geometries first introduced by F. Klein in 1871 and A. Cayley are number of geometries including Euclidean, Galilean, Minkowskian and Bolyai-Lobachevsikan [21]-[22]. Following Cayley and Klein, I. M. Yaglom distinguished these geometries with choosing one of three ways of measuring length (parabolic, elliptic or hyperbolic) between two points on a line and one of the three ways of measuring angles (parabolic, elliptic or hyperbolic) between two lines [18]. This gives nine ways of measuring lengths and angles and thus the nine plane geometries can be listed in Table 1. Many recent research are conducted in Cayley-Klein planes in terms of their group structure and group contraction. Also, their relationship between kinematic groups and concept of quantum kinematics is carried on [23]-[32].