Point Particle with Extrinsic Curvature as a Boundary of a Nambu–Goto String: Classical and Quantum Model

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Abstract. It is shown how a string living in a higher dimensional space can be approximated as a point particle with squared extrinsic curvature. We consider a generalized Howe–Tucker action for such a “rigid particle” and consider its classical equations of motion and constraints. We find that the algebra of the Dirac brackets between the dynamical variables associated with velocity and acceleration contains the spin tensor. After quantization, the corresponding operators can be represented by the Dirac matrices, projected onto the hypersurface that is orthogonal to the direction of momentum. A condition for the consistency of such a representation is that the states must satisfy the Dirac equation with a suitable effective mass. The Pauli–Lubanski vector composed with such projected Dirac matrices is equal to the Pauli–Lubanski vector composed with the usual, non projected, Dirac matrices, and its eigenvalues thus correspond to spin one half states.

1. Introduction

Extended objects, such as branes with extrinsic curvature are of great interest for physics [5, 20, 40, 45, 54, 56, 79–81]. A particular case is the point particle with extrinsic curvature, the so called “rigid particle” [3, 4, 28, 33, 39, 41, 42, 46–53, 57–59, 76–78]. Such an object, because of the second derivatives in the action, moves along a trajectory that is not a straight line, but a helix. The rectilinear component of the helical worldline corresponds to the particle’s momentum $p_\mu$, whereas the circular component is responsible for spin, $S_{\mu\nu}$. The quantities $p_\mu$, $S_{\mu\nu}$ and the orbital momentum $L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ satisfy the relations of a classical particle with spin [21, 22, 43, 44, 84, 90, 91]. A question arises as to whether the rigid particle can be a classical model for the quantum particle with spin, described by the Dirac equation. In fact, there are two types of rigid particles: those with the extrinsic curvature to

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the power one (type 1) \([3, 4, 33, 39, 41, 42, 45, 46, 48–50, 52–54, 56, 76–80]\), and
those with the squared intrinsic curvature (type 2) \([28, 33, 46, 47, 51, 57–59]\).

It was shown \([33, 46]\) that if one starts from an ordinary Nambu–Goto
string (without extrinsic curvature) living in a space with one extra space-
like dimension, then one can derive type 2 rigid particle as an approximation.
According to the authors of Ref. \([33, 46]\), such derivation was not quite consis-
tent. In Ref. \([58]\), it was shown how we can obtain the consistent rigid particle:
the squared extrinsic curvature with the correct sign in the rigid particle’s
action comes from a string living in a spacetime with an extra time-like
dimension. Starting from the Nambu–Goto string action, one can directly
arrive at the type 2 rigid particle action \([33, 46, 58]\). In this paper I will show,
following the previous work \([58]\), that if we start from the Polyakov form of
the string action, then as an intermediate approximation we obtain a generalized Howe–Tucker action that contains second order derivative.\(^1\) We will
study the classical and quantum equations of such a generalized Howe–Tucker
point particle action, describing what we will call type 2a rigid particle. The
system contains two first class and four second class constraints \([29, 30]\). The
Dirac brackets between the phase space variables associated with velocity
contains the spin tensor. The similar holds for the Dirac brackets associated
with acceleration. In the quantized theory, those Dirac bracket relations
become the commutation relations between the operators \([29, 30]\). It turns
out that these operators can be represented in terms of the gamma matrices,
multiplied by the generators of the Clifford algebra \(Cl(0, 2)\) of a 2-dimensional
space with signature \((-\cdot\cdot)\). The latter space is a subspace of the phase space
of our dynamical system. The signature \((-\cdot\cdot)\) comes from the space like type
of the chosen dynamical variables, the 4-acceleration and the projection of the
4-velocity onto a space like hypersurface. The Pauli–Lubanski vector turns
out to be the same as that for a Dirac particle. The analysis presented in
this paper thus leads to a conclusion that the type 2a rigid particle, upon
quantization, has spin \(\frac{1}{2}\). Deriglazov \([31]\) considered type 1 rigid particle,
whose dynamics is different, but he also found that the phase space vari-
ables can be quantized by gamma matrices and that the system has spin
one-half.

The physical states must satisfy the conditions imposed by the first
class constraints. This can be consistent with the second class constraints
and the representation of the operators in terms of the Clifford numbers if
we bring an additional time-like dimension into the game, besides the two
ones considered so far in our model.

In Sect. 2 we describe a scenario with an open string living in a \((D+1)\)-
dimensional target space whose \((D+1)\)th dimension, as well as the 1th one, are
time-like. For the \((D+1)\)th embedding function we choose \(X^{D+1}(\tau, \sigma) = \sigma\)
which in the considered scenario, illustrated in Fig. 1, is possible because of
the reparametrization invariance of the string action. We expand the
string embedding functions \(X^{\mu}(\tau, \sigma), \mu = 0, 1, 2, \ldots, D.\) into the Taylor series

\(^1\)If in the latter action we perform a further approximation, then we obtain the type 2
rigid particle action.